April 12

Cooperative Games

TU Games
Recall

Cooperative game

\[ \begin{array}{l}
1 \quad \text{TU}
\
\quad \text{transferable utility}
\
2 \quad \text{NTU}
\
\quad \text{nontransferable utility}
\end{array} \]
TU means players can make side payments

NTU: No side payments.

In a cooperative game, players can negotiate and reach an agreement.

→ best payoff for both players.
While negotiate threat counter threat

In TU we also need to decide side payments. First, what are the possible payoffs for players.

NITU feasible set
TU feasible set.
A bi-matrix game \((A, B)\) cooperative

Players may agree to achieve any payoff vector \((a_{ij}, b_{ij})\)
\[1 \leq i \leq n \quad 1 \leq j \leq n.\]

\(\implies\) \(m \cdot n\) points on a plane.

They can also agree to any probability mixture of these \(m \cdot n\) points.
The set of all such payoff vectors is the convex hull of these points.
This is the NTU feasible set.

Definition: The NTU feasible set is the convex hull of the m x n points \((a_{ij}, b_{ij})\) for \(1 \leq i \leq m\), \(1 \leq j \leq n\).
For TU games.

side payments

\((a_{ij}, b_{ij}) \rightarrow (a_{ij} + s, b_{ij} - s)\)

\(s > 0\) I pay I \(s\)

II transfers \(s\) utilities to I

\(s < 0\) I transfers \((-s)\) utilities to II.

\(\rightarrow\) more points can be achieved
\[(a_{ij} - s, b_{ij} + s)\] is the straight line passing \[(a_{ij}, b_{ij})\] with slope -1.

Definition: The TU feasible set is the convex hull of the set of vectors of the form \[(a_{ij} + s, b_{ij} - s)\] for \(1 \leq i \leq m, 1 \leq j \leq n\) and any real number \(s\).
Example

\[
\begin{pmatrix}
(4, 3) & (0, 0) \\
(2, 2) & (1, 4)
\end{pmatrix}
\]

NTU feasible set
TU feasible set

If we want to maximize both payoff → maximize \((x_1,y_1)\) in the feasible set
→ Upper right boundary

If an agreement is reached, no player can be made better off without making at least one other player worse off. → Pareto optimal points
Definition:
A feasible payoff vector \((v_1, v_2)\) is said to be Pareto optimal if the only feasible payoff vector \((v_1', v_2')\) such that \(v_1' \geq v_1, v_2' \geq v_2\) is the vector \((v_1', v_2') = (v_1, v_2)\)
TU game.

We imagine a negotiation before the game is played. Players try to decide a joint strategy, with possible side payments between players.

First the feasible payoffs they choose must be an Pareto optimal point.
Otherwise they will agree to move to a feasible point that is better for at least one of the players. It doesn’t hurt others.

They can also threat in order to get a better deal.

\[
\begin{pmatrix}
(5,3) \\
(0,0)
\end{pmatrix}
\begin{pmatrix}
(0,-4) \\
(3,6)
\end{pmatrix}
\]

First they can agree to \((3,6)\) the largest total payoffs
Player II then may argue that she should get at least 4½, that means only 1½ side payments to I. But I can threat. I should get at least 5. Otherwise he will change to Row I. This is a credible threat. Because II can not counter threat.
If I Row 1
(5, 3), (0, -4)

Counter threat should be
Col 2 $\rightarrow$ to make I bad

However in that case
1. 5 $\rightarrow$ 0
2. 3 $\rightarrow$ -4

II loses more
So II cannot threat I.
In general
threat
counter threat
counter-counter threat

So how to solve the TU problem?

Solution:

d) They agree that they should get the largest total payoff.
\[ \sigma = \max \max (a_{ij} + b_{ij}) \]

and then try to divide \( \sigma \)

\[ \text{Row } i \text{ in } A \text{ has } a_{i0} + b_{i0} = 0 \]

\((i_0, j_0)\) is the cooperative strategy

2) They need to agree on payoffs \((x^*, y^*)\)

\[ x^* + y^* = 0 \]
But $x^*$ may not be $A_{ij}$.
$y^*$ may not be $b_{ij}$, as they can make side payment.

Now imagine I and II have threat strategies.

If no agreement is reached,

$I \rightarrow \overline{p}$

$II \rightarrow \overline{q}$.
Then the payoff

\[ B = (D_1, D_2) \]

\[ = (\bar{p}^T A \bar{q}, \bar{p}^T B \bar{q}) \]

This is called the threat point.

Once they have the threat point, Player I will accept no less than \( D_1 \).

Player II will accept no less than \( D_2 \).

Since they can already
achieve these payoffs without reaching to an agreement.

→ They need to choose $(x^*, y^*)$ on $x + y = 6$

such that $x^* \geq D_1$
$y^* \geq D_2$

This is an interval $(D_1, 6-D_1)$

$(p_1, p_2)$

$(6-D_2, D_2)$
Note \( D_1 + D_2 \leq 6 \) because \((D_1, D_2)\)
is in the NTU feasible set and this set is convex.

Now the game is symmetric and has nothing to do with matrices \( A, B \).

\[ \rightarrow \] the mid point on the interval

\[ \overline{y} = (y_1, y_2) = \left( \frac{\sigma - D_1 + D_2}{2}, \frac{\sigma - D_1 + D_2}{2} \right) \]
\( \bar{y} = (y_1, y_2) \)

\( D = (0, 6-D_1) \)

\( D \subseteq (6-D, D_2) \)

\( \bar{y} \) is on the line passing \( D \) with slope 1.
3) Now how to choose \( \overrightarrow{D} = (D_1, D_2) \)?

I → maximize \( D_1 - D_2 \)

II → minimize \( D_1 - D_2 \)

Moreover

\[
D_1 - D_2 = \overrightarrow{P}^T A \overrightarrow{q} - \overrightarrow{P}^T B \overrightarrow{q} = \overrightarrow{P}^T (A - B) \overrightarrow{q}.
\]

→ zero-sum game with the matrix \( A - B \)

→ \( \xi = \text{UaC} (A - B) \)

\( \overrightarrow{p}^*, \overrightarrow{q}^* \) optimal
\( P^*, q^* \) are the best threat strategies.

\[
\vec{D}^* = (D_{1}^*, D_{2}^*)
\]

\[
= (P^* x^t A q^*, P^* x^t B q^*)
\]

\[
D_{1}^* - D_{2}^* = P^* x^t (A - B) q^*
\]

\[
= \nu a(A - B)
\]

\[
= \delta
\]

\[
\Rightarrow (x^*, y^*) = \left( \frac{6 + \delta}{2}, \frac{6 - \delta}{2} \right)
\]

\[
\Rightarrow \text{Solution. Side payment is } \frac{6 + \delta}{2} - a_{i,j}0.
\]
Example: TU game

\[
\begin{pmatrix}
(0,0) & (6,2) & (-1,2) \\
(4, -1) & (3,6) & (5,5)
\end{pmatrix}
\]

1) Max total payoff \((5,5)\)

\( l_0 = 2 \quad j_0 = 3 \)

\( 6 = 5 + 5 = l_0. \)

2) Side payment?

Solve the zero-sum game

\[
A - B = \begin{pmatrix}
0 & 4 & -3 \\
5 & -3 & 0
\end{pmatrix}
\]

Col 1 is dominate by Col 3
\[
\begin{pmatrix}
4 & -3 \\
-3 & 0
\end{pmatrix}
\]

No saddle point

\[\delta = \text{val} = \frac{-9}{4 - (-6)}\]

\[= - \frac{9}{10}\]

\[-3p = - \frac{9}{10}\]

\[p = 0.3 \quad 1 - p = 0.7\]

\[-3q = - \frac{9}{10}\]

\[q = 0.3 \quad 1 - q = 0.7\]

\[\overrightarrow{p}^* = (0.3, 0.7)^T \quad \text{threat strategies}\]

\[\overrightarrow{q}^* = (0, 0.3, 0.7)^T\]
\[
\bar{y}^* = \left( \frac{6+8}{2}, \frac{6-8}{2} \right)
\]

\[
= \left( \frac{14}{2}, \frac{-2}{2} \right)
\]

\[
\frac{10}{20} - 5 = \frac{9}{20}
\]

I makes side payment \( \frac{9}{20} \to II \)

The disagreement point
\[
\bar{D}^* = (D_1^*, D_2^*)
\]

\[
D_1^* = P^* \times \text{Area} \quad P^* = 3.41
\]

\[
D_2^* = P^* \times B \quad P^* = 4.31
\]