Apr 14

NTU

Nash Bargaining Model

The Lambda - Transfer approach
Recall TU game.

1) \( \delta = \max_i \max_j A_{ij} + b_{ij} \)
and \( \langle i_0, j_0 \rangle \)

2) Solve the zero-sum game \( A - B \)

\[ \Rightarrow \delta = \text{val} (A - B) \]

optimal strategy

\( \bar{p}, \bar{q} \)

\[ D_1 = \bar{p}^T A \bar{q} \]

\[ D_2 = \bar{p}^T B \bar{q} \]

\( D = (D_1, D_2) \) disagreement point.
\[ y^* = \left( \frac{6+8}{2}, \frac{6-8}{2} \right) \]

side payment = \frac{6+8}{2} - \text{aijo}

or \frac{6-8}{2} - \text{bijo}.

Example: More than one \((\text{aijo}, \text{bijo})\) have \(\text{aijo} + \text{bijo} = 0\)

\[
\begin{pmatrix}
11, 5 \\
8 \\
4, 2 \\
5, 0 \\
2, 3 \\
2, 1 \\
0, 0 \\
0, 1
\end{pmatrix}
\]
\[ 0 = 6 \cdot (1, 1) \cdot (2, 1) \]

More than one disagreement point.

\[
A - B = \begin{pmatrix}
-4 & 0 & -1 \\
2 & 1 & 1 \\
5 & -1 & 0
\end{pmatrix}
\]

Two saddle points.

Note that \( g = \text{Val}(A - B) \) is unique: \( g = 1 \)

But two sets of optimal strategies.
\[(2, 2)\]

\[D_1 = 1 \quad D_2 = 0\]

\[(2, 3)\]

\[D_1 = 2 \quad D_2 = 1\]

Two possible disagreement points

\[P = \left( \frac{\sigma + \delta}{2}, \frac{\sigma - \delta}{2} \right)\]

\[= \left( \frac{7}{2}, \frac{5}{2} \right) \text{ is unique.}\]
Choose \((1, 17)\)  
\((1, 5)\)

side payment \(\Pi \rightarrow I\)  
\[5 - \frac{5}{2} = \frac{5}{2}\]

Choose \((2, 1)\)  
\((4, 2)\)

side payment \(I \rightarrow \Pi\)  
\[4 - 3.5 = 0.5\]

1) There might be multiple disagreement points but they all give the same payoff vector \(\bar{v}\). 
2) There might be more than one choice of joint strategy. The side payment depends on which joint strategy is chosen.
NTU Games

Nash Bargain model

Recall the feasible set $S$ is the convex hull of $(a_{ij}, b_{ij})$ $1 \leq i \leq m$ $1 \leq j \leq n$

Use this approach a point $(u^*, v^*) \in S$, called the threat point, or status-quo point should be given.
Find the point \((\bar{u}, \bar{v}) \in S\) such that 

\((u - u^*)(v - v^*)\) is maximized at \((\bar{u}, \bar{v})\) over all points \((u, v) \in S\).

This \((\bar{u}, \bar{v})\) is the payoff vector that both players should agree.

Let's look at a more general Nash Bargaining model.
A Nash bargain model has two inputs

1) A bounded and closed, and convex set $S$ in the plane

2) a point $(u^*, v^*) \in S$, called threat point or status quo point in $S$

The output

$(\bar{u}, \bar{v}) = f(S, u^*, v^*)$

is considered as a fair solution of the bargain game.
It should satisfy the following axioms.

1) Feasibility $(\bar{u}, \bar{v}) \in S$

2) Pareto optimal If there is $(u, v) \in S$ s.t. $u \geq \bar{u}$, $v \geq \bar{v}$ then $(u, v) = (\bar{u}, \bar{v})$

3) Symmetry If $S$ is symmetric about the line $u = v$

and $u^* = v^*$

then $\bar{u} = \bar{v}$
4) Independence of irrelevant alternatives. If $T$ is a closed convex subset of $S$, then $f(T, u^*, v^*) = f(S, u^*, v^*)$.

5) Invariance under change of location and scale. If $T = \{ (u', v') : u' = \alpha_1 u + \beta_1, \quad v' = \alpha_2 u + \beta_2 \text{ for } (u, v) \in S \}$, where $\alpha_1, \alpha_2 > 0$, $\beta_1, \beta_2$ all constants, then

$$f(T, \alpha_1 u^* + \beta_1, \alpha_2 v^* + \beta_2) = (\alpha_1 \bar{u} + \beta_1, \alpha_2 \bar{v} + \beta_2)$$
Let's analyze the axioms.

1) is obvious. The final payoff should be feasible.

2) As we have talked before rational players should choose Pareto optimal points. As long as it doesn't hurt you, you should allow the other player to choose better payoff.
3) A symmetric game should have a symmetric solution.

4) This is controversial. It says if two players agree that \((\bar{u}, \bar{v})\) is a solution then the point far away from \((\bar{u}, \bar{v})\) and \((u^*, v^*)\) are irrelevant. So if we replace \(S\) by a smaller convex set \(T\).
that still contains \((\bar{u}, \bar{v})\) and \((u^*, v^*)\).

The solution is the same.

5) We have seen this before if we change the units of the game, or pay some money in advance. It should not change how the game is played. But not that here \(u' = \alpha_1 u + \beta_1\)

\[ v' = \alpha_2 v + \delta_2 \]
$u', v'$ can be changed independently. This is because we are studying NTU. We are assuming that the utilities can not be comparable.

Now the point is there is a unique point that satisfies all axioms. This is $(\bar{u}, \bar{v})$ that maximizes $(u-u^*)(v-v^*)$ among all $(u, v) \in S$. 
Theorem  There exists a unique function $f$ satisfying the Nash axioms. Moreover if there exists a point $(u, v) \in S$ such that $u > u^*$ and $v > v^*$ then $f(S, u^*, v^*)$ is the point $(\bar{u}, \bar{v}) \in S$ that maximizes $(u - u^*)(v - v^*)$ among points of $S$ such that $u \geq u^*, \ v \geq v^*$. 
Proof: First we check that the point \((\bar{u}, \bar{v})\) in
\[
S^+ = \{(u,v) \in S : u \geq h^*, \quad v \geq v^*\}
\]
indeed satisfies all Nash axioms.

1) \(\Rightarrow\) 4) obvious

For 5) The point is \(x(\bar{c}_0, \bar{c}_2, 0)
\)
\[(u - h^*)(v - v^*)\] is maximized over \(\bar{S}\) at \((\bar{u}, \bar{v})\)

\[\iff (x_1 u - x_1 u^*)(x_2 v - x_2 v^*)\]

is maximized over \(\bar{S}\) at \((\bar{u}, \bar{v})\)

\[\iff (x_1 u + \beta_1 - x_1 u^* - \beta_1)(x_2 v + \beta_2 - x_2 v^* - \beta_2)\]

is maximized over \(\bar{S}\) at \((\bar{u}, \bar{v})\)
\[
\begin{aligned}
  u' &= \alpha_1 u + \beta_1 \\
  v' &= \alpha_2 v + \beta_2
\end{aligned}
\]

\((u' - \alpha_1 u^* - \beta_1)(v' - \alpha_2 v^* - \beta_2)\)

is maximized over \(T^+\)

at \((\alpha_1 \bar{u} + \beta_1, \alpha_2 \bar{v} + \beta_2)\)

\(T^+ = \{(u',v')\mid u' = \alpha_1 u + \beta_1, v' = \alpha_2 v + \beta_2, \quad \text{for } (u,v) \in S^+\}\)

Secondly, we show that the axioms determine the point uniquely.
Note that if \((u^*, v^*) = (0, 0)\),

\(S\) is symmetric.

\[\text{(1) (2) (3) } \Rightarrow \text{. } u, v \text{ should be on the line } u = v\]

\(\text{and is the point furthest up the line.}\)

\(\text{(4) } \Rightarrow \text{ If } T \text{ convex closed }\)

\(T \text{ contained in } H_2 = \{ (u, v) : u + v \leq 2z \}\)

\(\text{for } z > 0.\)
and $(\alpha/\beta) \in T$

$(0, 0) \in T$

then $f(\mathbb{T}, 0, 0) = f(\mathbb{R}, 0, 0) = (\alpha/\beta)$

$R \subset H_2$ is some bounded symmetric set containing $T$
Now for a general convex set $S$ with $(u^*, v^*) \in S$
we can use transformation
to change it to a $T$ as above

Assume $(\hat{u}, \hat{v})$ is the point
of $S^*$ that maximize
$(u-u^*)(v-v^*)$ over $S^*$.

Define $\alpha_1, \beta_1, \alpha_2, \beta_2,$
$\alpha_1 \geq 0, \alpha_2 \geq 0.$

\[
\begin{align*}
\alpha_1 u^* + \beta_1 &= 0 & \hat{u} &> u^* \\
\alpha_1 \hat{u} + \beta_1 &= 1 \\
\alpha_2 v^* + \beta_2 &= 0 & \hat{v} &> v^* \\
\alpha_2 \hat{v} + \beta_2 &= 1
\end{align*}
\]
Then \( S \rightarrow T \) as above

As we just proved

\[(1, 1) = (\alpha_1 \hat{u} + \beta_1, \alpha_2 \hat{v} + \beta_2)\]

also maximise \( w \cdot v \) over \( T \).

Since the slope of \( w \cdot v = 1 \) at \((1, 1)\) is \(-1\)

\[T \text{ is contained inside } H_1\]

So \( f(T, 0, 0) = (1, 1) = (\alpha_1 \hat{u} + \beta_1, \alpha_2 \hat{v} + \beta_2)\)

By axiom 5, \( f(S \cdot n^+, v^*) \)

\[= (\hat{u}, \hat{v})\]
How to find this point $(\bar{u}, \bar{v})$ that maximize $(u-u^*)(v-v^*)$?

If the boundary of $T$ is smooth.
We need the tangent line of the boundary at \((\bar{u}, \bar{v})\)

\[
= \text{ the tangent line of } (u - u^*)(v - v^*) \text{ at } (\bar{u}, \bar{v})
\]

\[
f(x, y) = 0
\]

\[
d f = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = 0
\]

\[
\Rightarrow \text{ the slope of the tangent line}
\]

\[
\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}.
\]
\[ f(u, v) = (u-u^*)(v-v^*) - C \]

\[ \frac{\partial f}{\partial u} = v - v^* \]
\[ \frac{\partial f}{\partial v} = u - u^* \]

Therefore the slope is

\[ \left. \frac{dv}{du} \right|_{(u, v)} = - \frac{v - v^*}{u - u^*} \]

= - the line from \((u^*, v^*)\) to \((\overline{u}, \overline{v})\)

The angles are equal.
Example:

\( S = \) triangle with vertices \((0,0)\) \((0,1)\) \((3,0)\)

\((u^*, v^*) = (0,0)\)

\((\overline{u}, \overline{v})?)

\((1,0)\)

\((0,0)\) \((3,0)\)

slope \(\frac{1}{3}\) \(\text{and} \) slope \(-\frac{1}{3}\).

\((\overline{u}, \overline{v})\) is the intersection of

\(v = \frac{1}{3}u\) and \(v = -\frac{1}{3}u + 1\)

\((\overline{u}, \overline{v}) = \left( \frac{3}{2}, \frac{1}{2} \right)\)