April 19

NTU Game

Nash Bargaining model

\lambda - transfer
Recall
Nash Bargaining model
Given \((u^*, v^*) \in S\)
try to maximize
\((u - u^*)(v - v^*)\)
The slope of \((u - u^*)(v - v^*)\)
at a point \((\bar{u}, \bar{v})\)
is \[
\frac{dv}{du} = -\frac{\frac{\partial f}{\partial u}}{\frac{\partial f}{\partial v}} = -\frac{\bar{v} - v^*}{\bar{u} - u^*}
\]
2.

\[ S = \{ (x, y) \mid (x-2)^2 + 4(y-1)^2 \leq 8 \} \]

\((u^*, v^*) = (2, 1)\)

\((\bar{u}, \bar{v}) ?\)

The boundary is

\[ f(x, y) = (x-2)^2 + 4(y-1)^2 - 8 = 0 \]

\[ x \geq 2 \quad y \geq 1. \]

\[ \frac{\partial f}{\partial x} = 2(x-2) \quad \frac{\partial f}{\partial y} = 8(y-1) \]
\[
\frac{y-1}{x-2} = -\frac{d}{dx} = a \frac{\Delta x}{\Delta y} = \frac{2(x-2)}{8(y-1)}
\]

\[
\delta(y-1)^2 = 2(x-2)^2
\]

Moreover,

\[
(x-2)^2 + 4(y-1)^2 = 8
\]

\[
(x-2)^2 = 4 \\
(y-1)^2 = 1 \\
x \geq 2 \quad y \geq 1 \\
\overrightarrow{\text{u}} = (x, y) = (4, 2)
The previous example assumes the boundary is smooth, so the boundary is tangent with 
\[(u-u^*)(v-v^*) = c\]
for some \(c\).

What if the boundary is not smooth?
Example

NTU Game

\[
\begin{pmatrix}
(4, 3) & (0, 0) \\
(2, 2) & (1, 4)
\end{pmatrix}
\]

\[\left( u^*, u^* \right) = (0, 0) \quad \left( 1/4 \right)\]

The Pareto optimal boundary is
If we want the boundary tangent to \( uu = c \) for some \( c \).

The slope of the tangent line is \(-\frac{1}{3}\) \( v = -\frac{1}{3}u + \frac{13}{3} \).

The intersection of

\[
\begin{align*}
v &= \frac{1}{3}u \\
v &= -\frac{1}{3}u + \frac{13}{3}
\end{align*}
\]
The point is outside the feasible set \( S \).

That means in the interval

\[(1, 4) \rightarrow (4, 3)\]

\(uv\) is getting bigger as \(\rightarrow (4, 3)\).

Therefore \((\bar{u}, \bar{v}) = (4, 3)\)
The $\lambda$-transfer approach.

Axiom 4 is controversial.

Also we need additional input, the threat point $(w^+, v^+)$

NTU: utilities cannot be transferred.

But let's pretend that they can be transferred but there is an exchange rate $\lambda \leftarrow\text{unknown}$. $\lambda > 0$
That means:

Player I 1 unit

Player II 1 unit

For example I use $ US D

II uses RMB

I loses 1

\[ \Rightarrow II \text{ increase } \lambda = 6.5 \]

Then we pretend that we solve a TU game

with \((\lambda A, B)_{\lambda > 0}\)
\[ \sigma(\lambda) = \max_{i,j} \lambda a_{ij} + b_{ij} \]

\[ \delta(\lambda) = \text{val}(\lambda A - B) \]

The TU solution

\[ \left( \frac{\sigma(\lambda) + \delta(\lambda)}{2}, \frac{\sigma(\lambda) - \delta(\lambda)}{2} \right) \]

\[ \rightarrow \text{original game} \]

\[ \left( \frac{\sigma(x) + \delta(x)}{2\lambda}, \frac{\sigma(x) - \delta(x)}{2} \right) \]

Of course we don't know what \( \lambda \) is.
But if

$$\bar{y}(\lambda) = \left( \frac{\sigma(\lambda) + \delta(\lambda)}{2}, \frac{\sigma(\lambda) - \delta(\lambda)}{2} \right)$$

is in the NTU feasible set, then we don't need to make a utility transfer. We can use it as a NTU solution.

Now, there is a unique \( \lambda^* \), called \( \lambda^* \), such that \( \bar{y}(\lambda^*) \) is in the NTU
feasible set.
Then we use $\bar{F}(\lambda^*)$ as the NTU solution.
This $\lambda^*$ is called the equilibrium exchange rate.

How to find $\lambda^*$?
The general case is too complicated.
Simple case:

A and -B both have saddle points in the same position in the matrix.

In NTU game, this is called fixed threat point game.

For any \( \lambda \), \( \lambda A - B \) has a saddle point in the same position. This is the threat point.
Then we actually have two choices.

Knowing the threat point $(v^*, v^*) \rightarrow$ Nash model.

We will see that the solution given by the Nash model is the $\lambda$-transfer solution.
Example

\[
\begin{pmatrix}
-1 & 1 \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
1 & 3 \\
3 & -1
\end{pmatrix}
\]

\[
A = \begin{pmatrix}
-1 & 1 \\
0 & 3
\end{pmatrix}
\]

\[
-B = \begin{pmatrix}
-1 & -3 \\
0 & 1
\end{pmatrix}
\]

Saddle point

For any \( \lambda > 0 \)

\[
\lambda A - B = \begin{pmatrix}
-\lambda - 1 & \lambda - 3 \\
0 & 3\lambda - 1
\end{pmatrix}
\]

has a saddle point
So \((0,0)\) is the threat point \((u^*, v^*) = (0,0)\)

Recall the Nash solution

The slope of the tangent line at \((\bar{u}, \bar{v})\) is \(-\) slope of \((0,0)\) to \((\bar{u}, \bar{v})\)

\[= - \frac{\bar{v}}{\bar{u}}\]
\[ A \rightarrow \lambda A \]

means \( u \rightarrow \lambda u \)

Assume we scale by

\[ \lambda^* = \frac{\overline{u}}{u} \]

\[ (\overline{u}, \overline{v}) \rightarrow (\lambda^* \overline{u}, \overline{v}) \quad \parallel \]

\[ (\overline{v}, \overline{v}) \]

The slope of the tangent line becomes \(-1\)
Recall TU game

If \( D = (0,0) \)

the threat point

The TU-solution is

\[ \overline{y} = (\overline{u}, \overline{v}) \]

Scale back

\[ (\frac{\overline{u}}{\lambda^*}, \overline{v}) = (\overline{u}, \overline{v}) \]

\( \lambda \)-transfer solution = Nash solution
Moreover, the equilibrium exchange rate is $\lambda^* = \frac{u}{\overline{u}}$.

\[
\frac{3 - (-1)}{1 - 3} = \frac{4}{-2} = -2
\]

\[
\begin{cases}
  v = 2u \\
  v = -2u + 5
\end{cases}
\]

$(\overline{u}, \overline{v}) = (\frac{5}{4}, \frac{5}{2})$
Homework Problem.

§ 4. 6.

Find the NTU- solution and the equilibrium exchange rate of the following game without a fixed threat point

\[(a) \begin{pmatrix} (5, 2) & (0, 0) \\ (0, 0) & (1, 4) \end{pmatrix} \]
No saddle points

$S = NTU$ feasible set

We scale $u \rightarrow \lambda u \rightarrow 0$

or
Note the slope of $(\lambda/4)$ and $(5\lambda/2)$.

If slope $\geq -1$ then TU-solution outside $S_\lambda$. 

$S_\lambda$
If we want TU-solution in $S_\lambda$

$\rightarrow$ The slope from $(\lambda, 4)$ to $(5\lambda, 2)$ is $-1$

$$\frac{4-2}{\lambda-5\lambda} = -1 \quad \lambda = \frac{1}{2}$$
\[ \lambda = \frac{1}{2} \]

\[
(\lambda A, B) = \begin{pmatrix}
\left( \frac{5}{2}, 2 \right) & (0, 0) \\
(0, 0) & \left( \frac{1}{2}, 4 \right)
\end{pmatrix}
\]

\[ \theta = \frac{9}{2} \]

\[
\lambda A - B = \begin{pmatrix}
\frac{1}{2} & 0 \\
0 & -\frac{7}{2}
\end{pmatrix}
\]

Saddle point \( 0 \).

\[ \theta = 0. \]

\[ \overline{D} = (0, 0). \]

\[ \overline{x}_A = \left( \frac{9}{4}, \frac{9}{2} \right) \]

Rescale \( \overline{y} = \left( \frac{9}{2}, \frac{9}{4} \right) \)
Check \((\frac{9}{2}, \frac{9}{4})\) is in the NTU-feasible set.