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II 2 Matrix Games - Domination

We can write strategic
form \((X, Y, A)\) in a matrix

\[ X = \{ x_1, x_2, \ldots, x_m \} \]
\[ Y = \{ y_1, y_2, \ldots, y_n \} \]

\[ a_{ij} = A(x_i, y_j) \]

\[ A = \begin{pmatrix}
    a_{11} & a_{12} & \cdots & a_{1n} \\
    a_{21} & a_{22} & \cdots & a_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix} \]
I chooses a row
II chooses a column
II pays I $a_{ij}$

A mixed strategy for II is

$$\vec{p} = \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_m \end{pmatrix}$$

If II choose column $j$

The payoff is

$$\sum_{i=1}^{m} p_i a_{ij} = (p_1 \ldots P_m) \cdot \begin{pmatrix} a_{ij} \\ \vdots \\ a_{mj} \end{pmatrix}$$
For II

a mixed strategy

\[ \bar{q} = \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix} \]

The payoff against xi

\[ \sum_{j=1}^{n} a_{ij} q_j = (a_{i1}, a_{i2}, \ldots, a_{in}) \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix} \]

The payoff for \( \bar{p} \) against \( \bar{q} \)

\[ A(\bar{p}, \bar{q}) = \bar{p}^T A \cdot \bar{q} \]
2.1 Saddle points

Aij is called a saddle point if

(1) aij is the minimum of the ith row, aij ≤ aik ∀ k

(2) aij is the maximum of the jth column, aij ≥ aij ∀ l

If aij is a saddle point, the xi is optimal for I

yj is optimal for II

and aij is the value of the game
Example

\[ A = \begin{pmatrix}
  4 & 1 & -3 \\
  3 & 2 & 5 \\
  0 & 1 & 6
\end{pmatrix} \]

⊙ for minimum in a row
⊙ for maximum in a column

← Two circles mean a saddle point

You might have more than one saddle point, but the
values are the same

\[ a_{ij} \leq a_{il} \]
\[ \forall i \quad \forall j \]
\[ a_{ij} \geq a_{kl} \]

\[ \therefore a_{ij} \geq a_{kl} \]
\[ a_{ij} \leq a_{kl} \]

\[ \Rightarrow a_{ij} = a_{kl} \]
Another way to find saddle points:

row min

\[ A = \begin{pmatrix} 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & 0 & 2 & 1 \\ 3 & 1 & 2 & 2 \end{pmatrix} \]

col max 3 2 2 2

No match, no saddle points.

row min

\[ B = \begin{pmatrix} 3 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & 0 & 2 & 1 \\ 3 & 1 & 2 & 2 \end{pmatrix} \]

\[ B_{42} \text{ saddle point} \]
2x2 matrix game.

1. Test for a saddle point.
2. If there is no saddle point, solve by finding equalizing strategies.

The claim is if there is no saddle point, then there must exist equalizing strategy.
Assume there is no saddle point. \( q \perp q \)

\[
P A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]

Without loss of generality, assume \( a \geq b \)

Since \( b \) is not a saddle point
\( c > b \)

Since \( c \) is not a saddle point
\( c > d \)
\[
\Rightarrow a < a \Rightarrow a > b
\]
pay off for \( (p, 1-p) \)

\[
ap + d(1-p) \\
p \Rightarrow b_p + c(1-p)
\]

\[ap + d(1-p) = bp + c(1-p).\]
We can solve for $p$

$$\alpha p + d(1-p) = b p + c(1-p)$$

$$[(a-b) + (c-d)]p = c-d$$

$$p = \frac{c-d}{(a-b) + (c-d)}$$

$$0 < p < 1$$

Value

$$v = \alpha p + d(1-p)$$

$$= \frac{ac-bd}{(a-b) + (c-d)}$$
\[
A = \begin{pmatrix}
9 & 1 - q \\
-2 & 3 \\
3 & -4
\end{pmatrix}^{1 - p}
\]

No saddle points

\[
p = \frac{3 - (-4)}{(3 - (-2)) + (3 - (-4))}
\]

\[
p = \frac{7}{12}
\]

\[
q = \frac{3 - (-4)}{(3 - (-4)) + (3 - (-2))}
\]

\[
q = \frac{7}{12}
\]

\[
v = \frac{\text{det}}{(-2 - 3) + (-4 - 3)} = \frac{1}{12}
\]
Dominated strategy.

For $I$ strategy $x_i$ dominates strategy $x_k$ if the payoffs are always greater.

For $II$ strategy $y_i$ dominates strategy $y_j$ if the payoffs are less than or equal to owe money.
Definition.

We say the $i$th row of a matrix $A = (a_{ij})$ dominates the $k$th row if $a_{ij} \geq a_{kj}$ for all $j$.

We say the $i$th row strictly dominates the $k$th row if $a_{ij} > a_{kj}$ for all $j$. 
Similarly, the jth column of A dominates (strictly dominates) the kth column if 

\[ a_{ij} \leq a_{ik} \text{ (resp. } a_{ij} < a_{ik} \text{)} \]

for all i.

\[ X_i \text{ dominates } X_k \]

We also say \( X_k \) is dominated by \( X_i \).

If a strategy \( X \) is dominated by some other strategy, we call it dominated.
The point is we can remove dominated strategies. What I can achieve by choosing a dominated strategy $x$ can be achieved at least as well by using the strategy that dominates $x$. So delete $x$. 
More precisely,

1. Removal of a dominated strategy does not change the value of the game.

2. Removal of a strictly dominated strategy does not change the set of optimal strategies.
Example

\[
A = \begin{pmatrix}
2 & 0 & 4 \\
1 & 2 & 3 \\
4 & 1 & 2
\end{pmatrix}
\]

\[2 < 3\]

3rd column is dominated

\[
\rightarrow \begin{pmatrix}
2 & 0 \\
1 & 2 \\
4 & 1
\end{pmatrix}
\]

1st row is dominated

\[
\rightarrow \begin{pmatrix}
9 & 1-7 \\
1 & 2 \\
4 & 1
\end{pmatrix}
\]

\[
p \quad 1-p
\]
\[ p + 4(1-p) = 2p + (1-p) \]

\[ p = \frac{3}{4} \quad 1-p = \frac{1}{4} \]

\[ v = \frac{2}{4} \]

\[ q - + 2(1-q) = 4q + (1-q) \]

\[ q = \frac{1}{4} \]

*It is possible that a pure strategy is dominated by a mixed strategy.*
\[
A = \begin{pmatrix}
0 & 4 & 6 \\
5 & 7 & 4 \\
9 & 6 & 3
\end{pmatrix}
\]

2 \geq \frac{1}{2} (1 + 3)

2nd column is dominated by

\[
\left( \frac{1}{2}, 0, \frac{1}{2} \right)
\]

\[\rightarrow \quad \begin{pmatrix}
0 & 6 \\
5 & 4 \\
9 & 3
\end{pmatrix}
\]

2 \leq \frac{1}{3} 1 + \frac{2}{3} 3

\[\rightarrow \quad \begin{pmatrix}
0 & 6 \\
9 & 3
\end{pmatrix}\]
2.4 If the matrix is \( m \times 2 \) or \( 2 \times n \), we can draw graphs.

\[
\begin{pmatrix}
1 & 2 & 3 & 4 \\
-1 & 2 & 3 & 1 & 5 \\
-1 & 4 & 1 & 6 & 0 \\
\end{pmatrix}
\]

4 lines:

\[
2p + (1-t)4 \\
3p + (-p)1 \\
1p + 6(-p) \]

\[
5p + 0(-p) 
\]
Draw the lower boundary

Find the highest point
(intersection of 2 and 3)

\[3p + (1 - p) = p + 6(1 - p)\]

\[p = \frac{5}{7} \quad u = \frac{12}{7}\]
\[
\begin{bmatrix}
3 & 1 \\
1 & 6 \\
\end{bmatrix}
\]

for \( II \ (0, \frac{5}{7}, \frac{2}{7}, 0) \)

\( m \times 2 \) example

\[
\begin{bmatrix}
9 & 1 - q \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 5 \\
4 & 4 \\
6 & 2 \\
\end{bmatrix}
\]