Feb 23

$m \times 2$ example

\[
\begin{pmatrix}
9 & 1-q \\
4 & 4 \\
6 & 2
\end{pmatrix}
\]
v = 4, \text{ I plays Row 2}

3. The Principle of Indifference

The Equilibrium Thm

Application to 4 cases

3.2 Nonsingular matrices

3.3 Diagonal matrices

3.4 Triangular matrices

3.5 Symmetric games
A matrix game with \( m \times n \) matrix \( A \).

\[
\overline{p} = (p_1, \ldots, p_m)^T
\]

a mixed strategy for II.

Recall \( \overline{p} \) is an optimal strategy (or minimax strategy) for II means the payoff is at least \( V \) for every strategy \( j \) played by II.

\[
\sum_{i=1}^{m} p_i a_{ij} \geq V \text{ for all } j = 1, \ldots, n
\]
Similarly, a strategy 
\[ \overline{q}_t = (q_{11}, \ldots, q_{1n})^T \]
is optimal for II means
\[ \sum_{j=1}^{n} a_{ij} q_{ij} \leq V \]
for every i.

When both players play optimal strategies, the payoff is exactly V.

\[ V = (\sum_{j=1}^{n} b_j) V = \sum_{j=1}^{n} b_j V \leq \sum_{j=1}^{n} \overline{q}_j (\sum_{i=1}^{m} p_i a_{ij}) \]
\[ = \sum_{i=1}^{m} \sum_{j=1}^{n} p_i a_{ij} \overline{q}_j \]
\[= \sum_{i=1}^{m} P_i \left( \sum_{j=1}^{m} a_{ij} q_j \right) \]

\[\leq \sum_{i=1}^{m} P_i \cdot V = V \]

\[\Rightarrow \text{ The payoff } A(\bar{p}, \bar{q}) = \sum_{ij} P_i a_{ij} q_j \]

\[= V \]

We can also get each
\[\sum P_i a_{ij} \text{ with positive weight } q_j \text{ is } V \]

each \[\sum a_{ij} q_j \text{ with positive weight } P_i \text{ is also } V \]
Theorem (The Equilibrium theorem)

Consider a matrix game with \( m \times n \) matrix \( A \) and value \( V \).

Let \( \overrightarrow{p} = (p_1, \ldots, p_m)^T \) be optimal for I

Let \( \overrightarrow{q} = (q_1, \ldots, q_n)^T \) be optimal for II

Then

1. \( \sum_{j=1}^{n} a_{ij} q_j = V \) for all \( i \) with \( p_i > 0 \)

2. \( \sum_{i=1}^{m} p_i a_{ij} = V \) for all \( j \) with \( q_j > 0 \)
Proof. It suffices to prove (1).

We already know that

$$\sum_{j=1}^{n} a_{ij} q_{j} \leq V \text{ for all } i$$

Suppose for $p_k > 0$,

$$\sum_{j=1}^{n} a_{kj} q_{j} < V$$

We would have

$$V = \sum_{i,j} p_{i} a_{ij} q_{j} = \sum_{i=1}^{m} p_{i} \left( \sum_{j=1}^{n} a_{ij} q_{j} \right)$$

$$< \sum_{i=1}^{m} p_{i} V = V$$

$\square$
Note the condition $p_i > 0$  
$q_j > 0$

If row $i$ has positive weight  
$p_i > 0$ in the optimal strategy $\bar{x}$  
we call $i$ active.

So optimal for $I \Rightarrow$  
indifferent to active strategies of $II$  
optimal for $II \Rightarrow$  
indifferent to active strategies of $I$.

This is called the Principle of Indifference.
Example: Odd-or-Even.

Both players simultaneously call out one of \{0, 1, 2\}.

The matrix is

\[
\begin{pmatrix}
0 & 1 & -2 \\
1 & -2 & 3 \\
-2 & 3 & -4
\end{pmatrix}
\]

Assume X Even will play all 3 numbers, \( q_1 > 0, q_2 > 0, q_3 > 0 \).

We have
\[ p_2 - 2p_3 = V \]  \( \text{①} \)

\[ p_1 - 2p_2 + 3p_3 = V \] \( \text{②} \)

\[-2p_1 + 3p_2 - 4p_3 = V \] \( \text{③} \)

\[ p_1 + p_2 + p_3 = 1 \]

\[ ① + 2 \times ② + ③ \]

\[ \Rightarrow V = 0 \]

\[ \begin{cases} p_1 = \frac{1}{4} \\ p_2 = \frac{1}{2} \\ p_3 = \frac{1}{4} \end{cases} \]

\[ \Rightarrow \text{The same equations for } q \]

\[ \begin{cases} q_1 = \frac{1}{4} \\ q_2 = \frac{1}{2} \\ q_3 = \frac{1}{4} \end{cases} \]
The assumption \( x \) is correct.

\[
V = 0 \\
\bar{p} = ( \frac{1}{4}, \frac{1}{2}, \frac{1}{4} )^T \\
\bar{q} = ( \frac{1}{4}, \frac{1}{2}, \frac{1}{4} )^T.
\]

3.2 Nonsingular game.

If a \( m \times m \) nonsingular square matrix.

Assume 1 has an optimal strategy \( \bar{p} \) with \( p_i > 0 \) for all \( i \).

\[
\Rightarrow \sum_{j=1}^{m} a_{ij} q_j = V \quad \text{for all } i.
\]
\[ A \hat{q} = V \hat{1} \]
\[ \hat{1} = (1, \ldots, 1) \]
\[ \hat{q} = A^{-1} V \hat{1} \]

\( V \neq 0 \) since \( A \) nonsingular \( \hat{q} \neq 0 \)

Now \[ \sum_{j=1}^{m} q_j = 1 \]

\[ \Rightarrow 1 = \hat{1}^T \hat{q} \]
\[ = \hat{1}^T A^{-1} V \hat{1} \]
\[ V = \frac{1}{\hat{1}^T A^{-1} \hat{1}} \]
\[ \hat{q} = \frac{1}{\hat{1}^T A^{-1} \hat{1}} \cdot A^{-1} \hat{1} \]
For the same reason

\[ \hat{\mathbf{p}}^T = \frac{1}{\hat{\mathbf{1}}^T \hat{\mathbf{A}}^{-1}} \hat{\mathbf{1}}^T \hat{\mathbf{A}}^{-1} \]

Note that you need to check the conditions

\[ p_i \geq 0 \quad \text{for every } i \]

\[ q_{ij} \geq 0 \]

It turns out

Here \( \geq 0 \) is OK. You don't need \( p_i > 0 \) \( q_{ij} > 0 \).
Assume the square matrix $A$ is nonsingular, and $\overrightarrow{I}^TA^{-1}\overrightarrow{I} \neq 0$. Then the value

$$V = \frac{1}{\overrightarrow{I}^TA^{-1}\overrightarrow{I}}$$

and optimal

$$\overrightarrow{p}^* = V \overrightarrow{I}^TA^{-1}$$
$$\overrightarrow{q}^* = VA^{-1}\overrightarrow{I}$$

provided $\overrightarrow{p}^* \geq 0$ $\overrightarrow{q}^* \geq 0$.

A variant: $A$ singular but the addition of a positive constant to all entries of $A \to$ nonsingular. Apply the Theorem.
Odd or Even

\[
\begin{pmatrix}
0 & 1 & -2 \\
1 & -2 & 3 \\
-2 & 3 & -4
\end{pmatrix}
\]

singular

However +1

\[
A = \begin{pmatrix}
1 & 2 & -1 \\
2 & -1 & 4 \\
-1 & 4 & -3
\end{pmatrix}
\]

non-singular

\[
A^{-1} = \frac{1}{16} \begin{pmatrix}
13 & -2 & -7 \\
-2 & 4 & 6 \\
-7 & 6 & 5
\end{pmatrix}
\]

\[
\Rightarrow \quad V^1 = 1
\]

\[
\mathbf{F} = \left( \frac{1}{4}, \frac{1}{2}, \frac{1}{4} \right)^T
\]

\[
\mathbf{F} = \left( \frac{1}{4}, \frac{1}{2}, \frac{1}{4} \right)
\]
To answer the original question
\[ V = V' - 1 = 0 \]

3.3 Diagonal

If \( A \) is square and diagonal

\[ A = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_m \end{pmatrix} \]

Assume all \( d_i > 0 \)

Otherwise we would have dominated strategies H/W)
\[ p_i = \frac{V}{d_i} \]

\[ \sum p_i = 1 \]

\[ \Rightarrow \quad V = \left( \sum_{i=1}^{m} \frac{1}{d_i} \right)^{-1} \]

\[ q_i = \frac{V}{d_i} \]

**Example**

\[ C = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & 4 \end{pmatrix} \]

\[ V = \left( 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \right)^{-1} \]

\[ = \frac{12}{25} \]

\[ \overrightarrow{p} = \overrightarrow{q} = \left( \frac{12}{25}, \frac{6}{25}, \frac{4}{25}, \frac{3}{25} \right)^T \]
3.4 Triangular Games.

\[ T = \begin{pmatrix} 1 & -2 & 3 & -4 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \]

Assume all \( j \) for II active.

\[ p_1 = v \]

\[ -2p_1 + p_2 = v \]

\[ 3p_1 - 2p_2 + p_3 = v \]

\[ -4p_1 + 3p_2 - 2p_3 + p_4 = v \]

\[ \Rightarrow p_1 = v \quad p_2 = 3v \]

\[ p_3 = 4v \quad p_4 = 4v \]
Since \( \sum p_i = 1 \)
\[
12 V = 1 \\
V = \frac{1}{12}
\]

\( \bar{p} = \left( \frac{1}{12}, 4, \frac{1}{3}, \frac{1}{3} \right)^T \)

For \( \bar{q} \)
\[
q_1 - 2q_2 + 3q_3 - 4q_4 = V \\
q_2 - 2q_3 + 3q_4 = V \\
q_3 - 2q_4 = V \\
q_4 = V
\]

\( \bar{q} = \left( \frac{1}{3}, \frac{1}{3}, 4, \frac{1}{12} \right)^T \)

all \( p_i > 0 \) \( q_j > 0 \)