3.5 Symmetric Game

Symmetric means the two players both have the same set of strategies.

\[ a_{ij} = A(i,j) = -A(j,i) = -a_{ji} \]

\[ A = -A^T \]

The matrix is skew-symmetric!
Def: A finite matrix game is said to be symmetric if its game matrix is square and skew-symmetric.

Theorem: A finite symmetric game has value zero. Any strategy optimal for one player is also optimal for the other.

Proof: Assume \( \hat{F} \) is optimal for I. If II also uses \( \hat{F} \) the payoff would be zero, because
\[ \bar{p}^T A \bar{p} = (\bar{p}^T A \bar{p})^T \]
\[ = \bar{p}^T A^T \bar{p} \]
\[ = -\bar{p}^T A \bar{p} \]

\[ \Rightarrow \bar{p}^T A^T \bar{p} = 0 \]

This implies \( 0 \geq V \).

Apply the same argument for \( \Pi \).

\[ \Rightarrow \ V \geq 0 \]

\[ \Rightarrow \ V = 0. \]

Now if \( \bar{p} \) optimal for \( \Pi \)

\[ \sum_{i=1}^{m} p_i a_{ij} \geq 0 \text{ for } j \]

Then

\[ \sum_{j=1}^{m} a_{ij} p_j = -\sum_{j=1}^{m} p_j a_{ji} \leq 0 \]
for all i \\
\implies p \text{ is also optimal for } II \\
For the same reason. \\
If \overline{q} \text{ optimal for } II, \\
\overline{q} \text{ is optimal for } I \square \\

Example \\
Mendelssohn Games. \\
Both players choose an integer between 1 and 100. If the numbers are equal, A = 0
If the number is one larger than that chosen by the opponent, win 2.

If the number is two or more larger than that chosen by the opponent, loses 2.

The matrix is

\[
\begin{array}{ccccc}
 & 1 & 2 & 3 & 4 & 5 \\
\hline
1 & & 0 & -1 & 2 & 2 & 2 \\
2 & 1 & 0 & -1 & 2 & 2 \\
3 & -2 & 1 & 0 & -1 & 2 \\
4 & -2 & -2 & 1 & 0 & -1 \\
\end{array}
\]
\[ V = 0 \]

1 dominates 4, 5, 6, ...

Delete

\[
\begin{pmatrix}
  0 & -1 & 2 \\
  1 & 0 & -1 \\
 -2 & 1 & 0
\end{pmatrix}
\]

Assume \( q_1 > 0 \), \( q_2 > 0 \), \( q_3 > 0 \)

\[
\begin{align*}
  p_2 - 2p_3 &= 0 \\
  -p_1 + p_3 &= 0 \\
  2p_1 - p_2 &= 0
\end{align*}
\]

\[ p_1 + p_2 + p_3 = 1 \]

\[ p_1 = \frac{1}{4}, \quad p_2 = \frac{1}{2}, \quad p_3 = \frac{1}{4} \]

\[ \bar{p} = \bar{q} = (\frac{1}{4}, \frac{1}{2}, \frac{1}{4}, 0, \ldots) \]
4 Solving Finite Games

Best Response

Upper and lower Value

The Minimax Thm
\[(X, Y, A)\]

\[X = \{1, \ldots, n\}\]
Set of pure strategies for I

\[Y = \{1, \ldots, n\}\]
Set of pure strategies for II

Similarly introduce

\[X^* = \text{set of mixed strategies for I}\]

\[= \{\bar{\mathbf{p}} = (p_1, \ldots, p_n)^T : p_i \geq 0, \quad \sum_{i=1}^{m} p_i = 1\}\]
\[ Y^* = \{ \tilde{q} = (q_1, \ldots, q_n)^T; \]
\[ \tilde{q}_j > 0 \quad \forall j \]
\[ \sum_{j=1}^{n} q_j = 1 \}

You may say \( X \) is a subset of \( X^* \), \( Y \) is a subset of \( Y^* \).

For \( \tilde{p} \in X^* \), \( \tilde{q} \in Y^* \)

\[ A(\tilde{p}, \tilde{q}) = \tilde{p}^T A \tilde{q} \]
Suppose II announces that he is going to use \( \hat{q} \) to \( Y^* \).
The I should choose row \( i \) that maximizes
\[
\sum_{j=1}^{n} a_{ij} q_j = (A \hat{q})_i
\]
or equivalently, choose \( \hat{p} \in X \) that maximizes
\[
\hat{p}^T A \hat{q}.
\]
The point is

$$\max_{1 \leq i \leq m} \sum_{j=1}^{n} a_{ij} \varphi_j = \max_{x \in \mathcal{X}} \tilde{p}^T A \tilde{q}$$

Also, the active strategies in $\tilde{p}$ should achieve

$$\max_{1 \leq i \leq m} \sum_{j=1}^{n} a_{ij} \varphi_j$$

On one hand, we have "\( \leq \)"

because all pure strategies are in $\mathcal{X}^*$. 
On the other hand, we have "$\geq"$ because\( P \bar{A} \bar{q} \) is a weighted average of\( \sum_{j=1}^{n} a_{ij} q_j \) for\( i \) such that \( P_i > 0 \).

So the weighted average\( \bar{s} \leq \max \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} q_j \)

Also, if the weighted average\( \frac{1}{m} \sum_{i=1}^{m} P_i (\sum_{j=1}^{n} a_{ij} q_j) = \max \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} q_j \)
the every active strategy \( \bar{q} \) must have payoff

\[
\max_{1 \leq i \leq m} \sum_{j=1}^{n} a_{ij} \bar{q}_j
\]

Any \( \bar{q} \in X^* \) that achieves

\[
\max_{\bar{q} \in X^*} \bar{F}(\bar{q})
\]

is called a **best response** or a **Bayes strategy against** \( \bar{q} \).

In particular any row \( i \) that achieves

\[
\max_{1 \leq i \leq m} \sum_{j=1}^{n} a_{ij} \bar{q}_j
\]
is a pure Bayes strategy against \( \bar{q} \).

So every active strategy in a Bayes strategy is a pure Bayes strategy.

For \( \bar{q} \in Y^* \) in a finite game, there always exist pure Bayes strategies against \( \bar{q} \).
Similarly fix \( \overline{p} \in X^* \), for \( \Pi \) we want \( \min \):

\[
\min \sum_{j=1}^{m} p_i a_{ij} = \min_{\overline{q} \in Y^*} \overline{p}^T A \overline{q}
\]

Any \( \overline{q} \in Y^* \) achieves

is called a **best response**
or a **Bayes strategy against** \( \overline{p} \)
Now again II announces $\bar{q} \in Y^*$ first, then I of course would choose a Bayes strategy against $\bar{q}$, and II would lose

$$\max \sum_{1 \leq i \leq m} A_{ij} q_j^{ij}$$

For II minimize the loss choose $q$ so that

$$\max \sum_{1 \leq i \leq m} A_{ij} q_j$$ is min
\[ \overline{V} = \min_{\tilde{f} \in \mathcal{Y}^*} \max_{1 \leq i \leq m} \sum_{j=1}^{n} a_{ij} \tilde{q}_j \]

\[ = \min_{\tilde{f} \in \mathcal{Y}^*} \max_{\tilde{f} \in \mathcal{X}^*} \tilde{f}^T A \overline{q} \]

\( \overline{V} \) is the upper value of the game.

Any \( \tilde{f} \) achieves \( \overline{V} \) is called a minimax strategy.

\( \overline{V} \) is the smallest loss \( II \) can assure no matter what \( I \) does.
Similarly, if I announce \( \bar{p} \in X^* \), it would choose a Bayesian strategy:

\[
\rightarrow \quad \min_{1 \leq j \leq n} \sum_{i=1}^{m} p_i \alpha_{ij}
\]

\[
\rightarrow \quad I \text{ wants to maximize }
\]

\[
V = \max_{\bar{p} \in X^*} \min_{1 \leq j \leq n} \sum_{i=1}^{m} p_i \alpha_{ij}
\]

\[
= \max_{\bar{p} \in X^*} \min_{\bar{q} \in Y^*} \bar{p}^T \bar{A} \bar{q}
\]

\( V \) is called the lower value of the game.
Any $f$ that achieves $V$ is called a minimax strategy for $I$.

It is easy to see $V \leq \overline{V}$ always.

If $V = \overline{V}$ we say the value of the game exists and call the common value, the value of the game, and denote it by $V$. 
If the value of the game exists we refer to minimax strategies as optimal strategies.

The Minimax thm:
Every finite game has a value, and both players have minimax strategies.

Note that for $I$

\[
\text{minimax} \neq \bigwedge_{j} \left( \min_{i} a_{ij} \right) = v
\]
Instead it means
\[
\sum_{i=1}^{m} \pi_i a_{ij} \geq V \text{ for every } j.
\]

For II minimax means
\[
\sum_{j=1}^{n} a_{ij} q_j \leq V \text{ for every } i.
\]

For example, Homework

\[
\begin{pmatrix}
2 & 2 & 1 \\
1 & 0 & 1 \\
3 & 0 & 1
\end{pmatrix}
\]

saddle point

saddle means \( V = 1 \)
row 2 is minimax for I
column 3 is minimax for II

However,
against row 1, \(-1 < V\)
against row 2, \(1 = V\)
against row 3, \(1 = V\)

For a matrix game
\((A, X, Y)\)

We can add a constant to each entry or multiply each entry by a
positive constant,
it doesn't change the
optimal strategies.

If \( A' = (a_{ij}') \)
\[ a_{ij}' = a_{ij} + b \]

That means II pays I the
amount \( b \) before the game
starts.

If \( A' = (a_{ij}') \)
\[ a_{ij}' = c \cdot a_{ij} \quad c > 0 \]

Then you just change the scale
of the money.
Lemma. Let \( A = (a_{ij}) \) and \( A' = (a_{ij}') \) are matrices with \( a_{ij}' = c a_{ij} + b \) for \( c > 0 \).

Then the two matrix games have the same minimax strategies for I and II.

Also if \( V \) is the value of game \( A \), \( V' \) is the value of game \( A' \), then \( V' = c V + b \).