Feb 9  Green Hackenbush

on a general graph.

The Fusion Principle:
The vertices on any circuit may be fused without changing the Sprague-Grundy value of the graph.
1. Fuse two neighboring vertices by bringing them together to a single vertex and bending the edge joining them to a loop.

2. Replace a loop by a leaf.
Winning move?

1 is from 12

Chop one of the edge

\( \rightarrow \) value 0

Winning move
Part II

1. The strategic form of a Game.

We begin the study of two-person zero-sum games. Two players: Player I & Player II. Zero-sum: One's gain is the other's loss.
The gain of a player is measured by **payoff**.

This is a function associate to each outcome (both players played their strategies) a number.

**Definition (Strategic form)**

The strategic form, or normal form, of a two-person zero-sum game is given by a triple $(X, Y, A)$.
(1) \(X\) is a non-empty set.
the set of strategies of Player I

(2) \(Y\) is a non-empty set
the set of strategies of Player II

(3) \(A\) is a real-valued function
defined on \(X \times Y\).

It means the following:

Player I chooses \(x \in X\)
(a strategy), Player II
chooses \(y \in Y\). Then their
choices are made known and
I wins \(A(x, y)\) from II.
i.e. II wins - A(x,y)
from I (zero sum)

A(x,y) > 0  I win
A(x,y) < 0  II win
A(x,y) = 0  a tie

Note: A strategy ≠ A move.
In simple games, if both player only make one move
Yes a strategy = a move
In games like chess (You need to make several moves),
A strategy = what moves to make in every possible situation.

The program Deep Blue (1997) has only one strategy.

Google's AlphaGo is more complicated, has several strategies.
1.2 Example:
Odd or Even.
Two players, I and II, simultaneously call out one of the numbers one or two.
Play I  Odd: he wins if the sum of the numbers is odd.
Play II  Even: she wins if the sum of the numbers is even.
The amount paid to the winner from the loser is the sum of numbers in \$. So \( X = \{1, 2\} \)

\[ Y = \{1, 2\} \]

The strategic form is

\[
\begin{pmatrix}
\begin{array}{c}
\text{I} \\
\text{II (even)}
\end{array}
\end{pmatrix}
\begin{pmatrix}
\begin{array}{c}
9 \\
1
\end{array}
\end{pmatrix}
\]

\[
\begin{pmatrix}
\begin{array}{c}
\text{I (odd)} \\
\text{II (even)}
\end{array}
\end{pmatrix}
\begin{pmatrix}
\begin{array}{c}
1 \\
-2
\end{array}
\end{pmatrix}
\]

The numbers are payoffs \( A(x,y) \)
Analyse the game from 1's point of view.

If 1 calls "1" 3/5 out of the time, and "2" 2/5 out of the time at random.

Case 1, If II calls "1".
\[
\frac{3}{5} \times (-2) + \frac{2}{5} \times (3) = 0
\]

Case 2, If II calls "2".
\[
\frac{3}{5} \times (3) + \frac{2}{5} \times (-4) = \frac{1}{5}
\]
The means on average, I wins $\frac{1}{5}$ if II calls "2" and doesn't lose money otherwise.

How to maximize the average payoff?

The key idea is: the payoff should be indifferent to II's choice. No matter what strategy II chooses, the payoffs are the same.
This gives an equation
Assume the probability to
call "1" is \( p \). The probability
to call "2" is then \( 1 - p \).

If II calls "1"

\[
A ((p, 1-p), 1) = p(-2) + (1-p)(-3)
\]

If II calls "2"

\[
A ((p, 1-p), 2) = p(3) + (1-p)(-4)
\]

Let them be equal

\[
p(-2) + (1-p)(3) = p(3) + (1-p)(-4)
\]

\[
\Rightarrow \ p = \frac{7}{12}
\]
In this case, the common payoff is

\[ p(-2) + (1-p)3 \]

\[ = \frac{7}{12}(-2) + (\frac{5}{12})3 \]

\[ = \frac{1}{12} \]

This means on average, each time I plays the game, no matter what II does, I wins \( \frac{1}{12} \).

This strategy \((p, 1-p) = (\frac{7}{12}, \frac{5}{12})\) is called the equalizing strategy.
The average pay off $\frac{1}{12}$ for the equalizing strategy (the strategy of II becomes irrelevant) is called the value of the game.

For II, she can also ensure that the average loss is $\frac{1}{12}$.

Let $q$ be the probability that II calls "1". $1 - q$ is the probability that II calls "2".

If I calls "1" \( \checkmark \) I calls "2"

\[ (-2)q + 3 (-q) = 3q + (-4)(1 - q) \]

\[ q = \frac{7}{12}, \quad 1 - q = \frac{5}{12} \]
\[-2q + 3(1-q) = -2 \times \frac{7}{12} \]
\[+ 3 \times \frac{5}{12} \]
\[= \frac{1}{12} \]

The strategy that insures the return = the value of the game is called an optimal strategy or a minimax strategy.

In this example, the equalizing strategy is a minimax strategy. (This is the key idea)
It is useful to make a distinction between a pure strategy and a mixed strategy.

The elements of X or Y are called the pure strategies.

The mixture of pure strategies at random in various proportions is called a mixed strategy.

e.g. \( \left( \frac{5}{12}, \frac{7}{12} \right) \)

A pure strategy \( x \in X \) can be considered as the mixed strategy that chooses \( x \) with probability 1.
The Minimax Theorem.

Odd-or-Even is an example of a finite game.

A two-person zero-sum game \((X, Y, A)\) is said to be a finite game if both strategy sets \(X\) and \(Y\) are finite sets.
The Minimax Theorem:
For every finite two-person zero-sum game,
(1) there is a number \( V \), called the value of the game.
(2) there is a mixed strategy for Player I such that I's average gain is at least \( V \) no matter what II does.
(3) there is a mixed strategy for Player II such that II's average loss is at most \( V \) no matter what I does.
If $v = 0$, the game is fair.
If $v > 0$, the game favors I.
If $v < 0$, the game favors II.