Jan 14.

Introduction

We all know about games: chess, poker, baseball, computer games ...

In this class, "game" means something more specific.
They are math models of conflicts/ cooperations between rational players.

So first, we have players. We usually label them by numbers 1, 2, …, $n \leftarrow$ some integer.

More formally, we have a set of players

$N = \{1, \ldots, n\}$
A set is simply a collection of objects we want to study. It's just a convenient language.

We are most interested in \( n=2 \).

We say the players are rational. That means they will try to win the game, and they will make the optimal move (if available).
Secondly, as math models, the games have explicit rules. Each player can make certain moves. The moves of all players $\Rightarrow$ positions. If there is something not controlled by players, we can use probability theory to model it.
Finally, either we know how to win/lose a game (like a chess) or we can model the payoffs as taking on numerical values.

It turns out these abstract games have wide applications.
In economy, several firms (players) compete for the market can be regarded as a game.

In biology, the competitions between species | genes (players) can also be modeled as games.

Question: What are the payoffs for the biological games? (Population)
1.1 Take-away games.
Remove chips from a pile of 13 chips (or flags, "21 flags")

i) Two plays \( N = \{1, 2\} \).

ii) Moves: a player can remove 1 or 2 or 3 chips from the pile (He/She must remove at least one.)
iii) Player 1 starts. Then players alternate moves.

iv) The player that removes the last chip wins.
Backward induction:

If there are 1, 2, 3 chips left, the player who moves next wins.

If there are 4 chips left, loses.

If there are 5, 6, 7, make it 4 → wins.

If there are 8, loses.
What is special about the positions 0, 4, 8, 12?

The are target positions. We want to move into them so that the player who moves next loses.

If there are 13 chips, Player 1 removes one chip. Then guarantee the win! No matter what player 2 does, make the remaining divisible by 4.
Win the game by 4 moves.

This is an example of a impartial combinatorial game. A game is a combinatorial game if:

1) There are two players
   \[ N = \{1, 2\} \]

2) There is a set, usually finite, of possible positions of the game (e.g., 13)
3) The rules specify for both players and each position which moves are legal moves. (e.g. 1, 2, 3)

4) Player 1 starts. Then the players alternate moving.

5) The game ends when a position is reached from which no further moves are possible for the player whose turn it is to move.
Under normal play rule, the last player to move wins.

Under the misère play rule, the last player to move loses.

Why do I make the statement of §1 so complicated?

Because it is possible that for some position, two players have different options of legal
moves. In this case, the game is called partizan.

If both players always have the same set of moves for each position, the game is called impartial.

Important concept

P-position vs N-position
In the previous game

The target positions

0, 4, 8, \ldots

mean the position that
can guarantee a win for the

Previous player. We call
them P-positions.

1, 2, 3, 5, 6, \ldots are

winning for the Next player

We call them N-positions.
In impartial combinatorial games, one can find in principle P-positions and N-positions by backward induction.

Start from a terminal position meaning no further moves are possible.

Under the normal play rule

1) Every terminal position is a P-position

2) Label every position that can reach a P-position in one move
as an N-position.

Step 3: Find those positions whose only moves are to N-positions, label them as P-positions.

Step 4. No new P positions found in Step 3 stop;
otherwise return to Step 2.

Note that 2) and 3) are not symmetric
Because next player controls
the next move!
For the normal rule:

1) All terminal positions are $P$-positions
2) From an $N$-position there is at least one move to a $P$-position.
3) From a $P$-position, every move is to an $N$-position.
More general, given a pile of chips, each player can remove $s$ chips if $s$ is a positive integer from a set $S$. This set $S$ is called the subtraction set. The game is called the subtraction game with subtraction set $S$. How about 100 chips with $S = \{ 1, 3, 4 \}$?