Jan 21

1.3 Graph

In the previous lectures, we use drew graphs for the games: dots are positions, arrows are moves.

This is a very useful way to study games. Today we study graphs systematically. We also introduce Sprague-Grundy function.
Definition:
A directed graph, $G$ is a pair $(X, F)$ where $X$ is a nonempty set of vertices (positions) and $F$ is a function from $X$ to the set of subsets of $X$. For each $x \in X$, $F(x)$ is a subset of $X$, $F(x) \subseteq X$.

For a given $x \in X$, $F(x)$ represents the positions to which a player may move from $x$. 
(called the followers of x).

If F(x) is empty, x is called a terminal position.

This definition is equivalent to giving vertices and arrows connecting vertices. F(x) tells you you should draw an arrow from x to every element in F(x).
An impartial combinatorial game can be played on a graph $G = (X, F)$

1) Player 1 moves first starting from some vertex $x_0 \in X$

2) Players alternate moves.

3) At position $x$, the player whose turn it is to move chooses a position $y \in F(x)$

4) The player who is confronted with a terminal position at his turn loses.
Under the normal play rule, draw a graph:

A path of length $m$ is a sequence $x_0, x_1, x_2, \ldots, x_m$ such that $x_i \in F(x_{i-1})$ for all $i = 1, \ldots, m$.

A cycle is a path $x_0, x_1, \ldots, x_m$ with $x_0 = x_m$ and distinct $x_0, x_1, \ldots, x_{m-1}$.
It is possible that a graph game could continue for an infinite number of moves.

We can add conditions to the graph to avoid that. For example, for each $x_0 \in X$, there is a number $n$ (may depend on $x_0$), such that every path starting from $x_0$ has length $\leq n$. That means starting from $x_0$, we can always end the game within $n$ moves.
Such a game is called **progressively bounded**.

Note that if there is a circle, it is not possible.

**Example:**
A subtraction game with

\[ S = \{1, 2, 3\} \]

5 chips

\[ F(0) = \emptyset \quad \text{terminal} \]
\[ F(1) = \{ 0 \} \]
\[ F(2) = \{ 1, 0 \} \]
For all the rest \( k \geq 3 \)
\[ F(k) = \{ k-3, k-2, k-1 \} \]

3.2 The Sprague-Grundy Function.

Some more refined data than \( P \) - positions / \( N \) - positions
Definition (Important)
The Sprague–Grundy function of a graph $G = (X, F)$ is a function $g$, defined on $X$

$$g : X \to \mathbb{N} = \{0, 1, 2, \ldots \}$$

such that

$$g(x) = \min \{ n \geq 0, \ n \not\equiv g(y) \quad \text{for } y \in F(x) \}$$

1. You know the values on $g$ on $F(x)$ first, then you know $g(x)$. 
2) \( g(x) \) is the smallest non-negative integer NOT found among the Sprague-Grundy values of the followers of \( x \).

Define \( \text{mex} \) of a set \( S \) of non-negative integers as the smallest non-negative integer NOT in this set \( S \).

Then we have

\[
g(x) = \text{mex} \{ g(y) : y \notin F(x) \}
\]
e.g. \( \text{mex} \{1, 3, 4\} = 0 \).

0, 2, 5, 6, ... 

e.g. \( \text{mex} \{\text{even numbers} \} = 1 \)

1, 3, 5, ... 

For a graph, the Sprague-Grundy function \( g \) may not exist, but if there exists such a \( g \), it is unique.

If \( x \) is a terminal position

\[
F(x) = \emptyset, \\
\text{mex} \{F(x)\} = 0 \quad g(x) = 0
\]
Then \( g \) is determined recursively (backward induction).

We will give examples later.

The important thing is if \( g \) exists

\[
\begin{align*}
P\text{-positions} & : g(x) = 0 \\
N\text{-positions} & : g(x) \neq 0
\end{align*}
\]

Check the following

1) If \( x \) is a terminal position
   \[ g(x) = 0. \]

2) At a position \( x \), \( g(x) = 0 \), every follower \( y \in F(x) \) has \( g(y) \neq 0 \).

3) At a position \( x \), \( g(x) \neq 0 \),
there is at least one follower
\( y \in F(x) \) such that \( g(y) = 0 \).

Examples:

1. See the textbook
   Figure 3.2 Page 7-16

2. For the subtraction
   game with \( S = \{1, 2, 3\} \)

Draw the graph

\[ \text{Diagram of graph} \]
We know
0 terminal $F(0) = \emptyset$
  $g(0) = 0$
1, $F(1) = \{0\}$
  $g(1) = \max \{0\} = 1$
2, $F(2) = \{0, 1\}$
  $g(2) = \max \{0, 1\} = 2$
3, $F(3) = \{0, 1, 2\}$
  $g(3) = \max \{0, 1, 2\} = 3$
4, $F(4) = \{1, 2, 3\}$
  $g(4) = \max \{1, 2, 3\} = 0$
5, $F(5) = \{2, 3, 4\}$
  $g(5) = \max \{2, 3, 0\} = 1$
\[ x \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \]

\[ f(x) \quad 0 \quad 1 \quad 2 \quad 3 \quad 0 \quad 1 \quad 2 \quad 3 \]

So \[ g(x) = x \pmod{4} \]

\(\text{3} \quad \text{At least Half} \)

One pile game:

Remove at least half of the counters.

The only terminal position is 0

\[ 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \]

\[ 0 \quad 1 \quad 2 \quad 2 \quad 3 \quad 3 \quad 3 \quad 3 \]
So $F(x) = \left\lfloor \frac{x}{2} \right\rfloor$.

$g(x) = \min \{ k : 2^k > x \}$

4. Even every path has a finite length. The length of paths starting from $x_0$ can go to infinite.

$F(x)$ can be an infinite set $g(x)$ is not a finite number.

See Figure 3.3 Page 2-17
Here \( g(x) = \omega \uparrow \) a transfinite number. This is not a number, it is an ordinal, it means the first ordinal after all integers.

In this case there is a circle \( a, b, c, a \) e terminal \( g(e) = 0 \) d, \( F(d) = \{e\} \) g(d) = 1
How about c?

\[ F(c) = \{ a, d \} \]
\[ F(a) = \{ b \} \]
\[ F(b) = \{ c \} \]

Since \( F(b) \) has only one element

\[ g(b) = \begin{cases} 0 & \text{if } g(c) \neq 0 \\ 1 & \text{if } g(c) = 0 \end{cases} \]

Assume \( g(b) = 0 \)

\[ g(a) = 1 \]

\[ \Rightarrow g(c) = 0 \]

Assume \( g(b) = 1 \)

\[ g(a) = 0 \]

\[ \Rightarrow g(c) = 2 \]
So no Sprague-Grundy function exists.

In fact, if you face position $c$, you will not move to $d$ (an \( N \)-position).

You will move to $a$:

\[
    c \rightarrow a \rightarrow b \rightarrow c \rightarrow a \rightarrow \cdots
\]

repeat forever.

It is defined to be a tie.