Mar 15

Behavioral Strategies

Instead of randomizing strategies, we can simply randomizing moves for each information set.
For example, if the first move is the deal of one card from a deck of 52 cards to Player 1. Then after seeing the card, 2 either bets or passes. The Player II takes some actions.
\[
\frac{1}{52} \quad \frac{1}{52} \quad \frac{1}{52} \\
1 \quad b \quad b \quad b \quad \ldots \quad b \quad p
\]

52 information sets for I
Pure strategies \(2^{52}\)

\(\rightarrow\) mixed strategy
a vector of length \(2^{52}\)

If for each information set we assign the probability of playing \(p\)
We only need 52 numbers between 0 and 1. This is a much smaller subset called the set of behavioral strategies.

For example, 3 information set each having two edges $(a, b)$. 8 Pure strategies
For a general mixed strategy, the weights are independent.

Behavioral strategies for information set I

P_i to choose a
For information set 2

$P_2$ to choose $a$

For information set 3

$P_3$ to choose $a$.

$p_1 p_2 p_3 \rightarrow (a, a, a)$

$(1 - p_1) p_2 p_3 \rightarrow (b, a, a)$

$p_1 (1 - p_2) p_3 \rightarrow (a, b, a)$

$(1 - p_1) (1 - p_2) p_3 \rightarrow (b, b, a)$

$p_1 p_2 (1 - p_3) \rightarrow (a, a, b)$

$(1 - p_1) p_2 (1 - p_3) \rightarrow (b, a, b)$

$p_1 ((-r_2) (1 - p_3) \rightarrow (a, b, b)$

$(1 - p_1) (-r_2) (1 - p_3) \rightarrow (b, b, b)$

$(1 - p_1) (-r_2) (1 - p_3) \rightarrow (b, b, b)$

$(1 - p_1) (-r_2) (1 - p_3) \rightarrow (b, b, b)$
The weights have relations.
So behavioral strategies
are very special mixed strategies.

Is it sufficient to consider only behavioral strategies?

Yes, if both players can have perfect recall, i.e.
remember all past moves.
A Theorem by Kuhn.
Example:

This is not a game with perfect recall.
Pure strategies for I

\[(f, c) \quad (f, d) \quad (g, c) \quad (g, d)\]

for II \[a, b\]

\[
\begin{array}{cc}
(a, b) \\
(f, c) & 1 & -1 \\
(f, d) & 1 & 0 \\
g, c & 0 & 2 \\
g, d & -1 & 2 \\
\end{array}
\]

\[
\Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \quad \text{v} = \frac{2}{3}
\]
$$\overline{p} = (0, \frac{2}{3}, \frac{1}{3}, 0)^T$$

$$\overline{q} = (\frac{2}{3}, \frac{1}{3})^T$$

This mixed strategy \( \overline{p} \) is not a behavioral strategy.

For a behavioral strategy

\( \overline{p}_f \) for \( f \)

\( \overline{p}_c \) for \( c \).

\( (\overline{p}_f \overline{p}_c, \overline{p}_f (1-\overline{p}_c), (1-\overline{p}_f)\overline{p}_c, (1-\overline{p}_f)(1-\overline{p}_c)) \)
\[
\begin{cases}
P_t \cdot P_c = 0 & \text{(1)} \\
P_t \cdot (1 - P_c) = \frac{2}{3} & \text{(2)} \\
(1 - P_t) \cdot P_c = \frac{1}{3} & \text{(3)} \\
(1 - P_t) \cdot (1 - P_c) = 0 & \text{(4)}
\end{cases}
\]

\[
0 = (\text{(1)} \times \text{(4)}) = \frac{2}{3} \\
= (\text{(2)} \times \text{(3)}) = \frac{2}{9}
\]

Impossible!
Problem 3.

\[ A = \begin{bmatrix}
0 & -1 & -1 & -1 \\
1 & 0 & -1 & -1 \\
1 & 1 & 0 & -1 \\
\vdots & \vdots & \vdots & \vdots
\end{bmatrix} \]

We want to find out \( \overline{V} \) and \( \underline{V} \).

Recall the definition

\[ \overline{V} = \inf_{q \in Y^*} \sup_{p \in \mathcal{P} \times X^*} \overline{p}^T A q \]
\[
\bar{V} = \sup_{\overrightarrow{p} \in \mathcal{X}} \inf_{\overrightarrow{q} \in \mathcal{Y}} \overrightarrow{p}^T \overrightarrow{A} \overrightarrow{q}
\]

Fix $\overrightarrow{q} = (q_1, q_2, \ldots)$

\[
\sup_{\overrightarrow{p} \in \mathcal{X}} \overrightarrow{p}^T \overrightarrow{A} \overrightarrow{q}
\]

\[
= \sup_{i \geq 1} \sum_{j} A(i,j) q_j
\]

Let $a_i = \sum_{j=1}^{\infty} A(i,j) q_j$

\[
\{a_i\} \text{ is an increasing sequence.}
\]

(Row $i$ is dominated by Row $i+1$).

\[
A(i,j) \leq A((i+1),j)
\]

and $q_j \geq 0$. 
$$a_i \leq a_{i+1}$$

$$a_i = \frac{\sum_{j=1}^{\infty} A_{i,j} q_j}{q_j} = \sum_{j=1}^{\infty} q_j - \sum_{j=i+1}^{\infty} q_j$$

$$= 1 - \sum_{j=i}^{\infty} q_j - \sum_{j=i+1}^{\infty} q_j$$

Since $$q_j \geq 0 \quad \forall j$$

$$1 - 2 \sum_{j=i}^{\infty} q_j \leq a_i \leq 1$$

Since $$\sum_{j=1}^{\infty} q_j = 1$$

$$\sum_{j=i}^{\infty} q_j \to 0 \quad \text{as} \quad i \to \infty$$
\[ A \in \mathbb{N}, \quad s + i \geq N \]

\[ \sum_{j=i}^{\infty} q_j < 3. \]

\[ \Rightarrow \quad 1 - 2e \leq a_i \leq 1 \quad i > N \]

\[ \{a_i\} \text{ increasing and} \]

\[ \sup a_i = \lim_{i \to \infty} a_i = 1 \quad i \geq 1 \]

\[ \Rightarrow \quad \sup_{i \geq 1} \sum_{j=1}^{\infty} A(i,j)q_j = 1 \quad (a) \]

\[ \overline{V} = \inf_{\mathbf{q} \in \mathcal{Y}^k} \sup_{i \geq 1} \sum_{j=1}^{\infty} A(i,j)q_j \]

\[ = \inf_{\mathbf{b} \in \mathcal{Y}^k} 1 = 1 \quad (b) \]
The game is symmetric
(Recall the definition)
\[ A(i,j) = -A(j,i) \]
If II can assure the smallest
average loss 1, I can also
assure the smallest loss 1.
i.e. max gain -1.
\[ V = -1 \quad (c) \]
(This is obvious. The smallest
payoff \[ A(i,j) = -1 \] )
So any strategy is
\textit{minmax}

This example shows that when the value doesn't exist \( (V \neq \overline{V}) \)

The \textit{minimax} doesn't mean much.
6.2

Multi-stage games

We study a new type of game. Replace an entry in the matrix by a matrix.
$G_1 = \begin{pmatrix} 0 & 3 \\ 2 & -1 \end{pmatrix}$  
$G_2 = \begin{pmatrix} 0 & 1 \\ 4 & 3 \end{pmatrix}$

$$G = \begin{pmatrix} G_1 & 4 \\ 5 & G_2 \end{pmatrix}$$

Game $G$ is played as follows.

I chooses a row
II chooses a column

If I chooses Row 1, II chooses Column 1 $\implies$ play Game $G_1$;
If I chooses Row 2, II chooses Column 2 $\implies$ play Game $G_2$. 
To solve $G$, solve $G_1$, $G_2$ first. Then replace matrices by their values

$$G_1 = \begin{pmatrix} 0 & 3 \\ 2 & -1 \end{pmatrix}$$

Optimal for I $(\frac{1}{2}, \frac{1}{2})$
Optimal for II $(\frac{2}{3}, \frac{1}{3})$

$v_1 = 1$

$$G_2 = \begin{pmatrix} 0 & 1 \\ 4 & 3 \end{pmatrix}$$ saddle point
Optimal for I $(0, 1)$
Optimal for II $(0, 1)$

$v_2 = 3$
Replace A by

\[
\begin{pmatrix}
1 & 4 \\
5 & 3
\end{pmatrix}
\]

No saddle point

\[p + 5(1-p) = 4p + 3(1-p)\]
\[p = \frac{2}{5}\]

\[q + 4(1-q) = 5q + 3(1-q)\]
\[q = \frac{1}{5}\]

I \quad \left(\frac{3}{5}, \frac{3}{5}\right)

II \quad \left(\frac{1}{5}, \frac{4}{5}\right)

V = \frac{17}{5}
$\mathbf{A}$ is also equivalent to the following matrix game:

$$
\begin{pmatrix}
0 & 3 & 4 & 4 \\
2 & -1 & 4 & 4 \\
5 & 5 & 0 & 1 \\
5 & 5 & 4 & 3
\end{pmatrix}
$$

This is because we can consider the pure strategy of 2 \( \{(1,1), (1,2), (2,1), (2,2)\} \).

(iii) First choose Rowi in $\mathbf{A}$, then if play $A_i$, choose Rowj.
This is an example of a multi-stage game.

I, II. II is trying to perform some forbidden action in one of n time periods, and she must perform one. Player I is allowed to inspect II secretly just once in the n time period.
If II acts while I inspects, II loses 1.

If II acts I must inspect 0.

Let \( G_n \) denote the game.

\[ n = 1 \quad G_1 = (1) \]

If \( n > 1 \) then consider the first time period.

I can either inspect, or wait.

II can either act, or wait.
\[ G_n = \begin{pmatrix} \text{act} & \text{wait} \\ \text{wait} & G_{n-1} \end{pmatrix} \]

\[ G_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]

\[ \left( \frac{1}{2}, \frac{1}{2} \right) \left( \frac{1}{2}, \frac{1}{2} \right) \]

\[ \nu(G_2) = \frac{1}{2}. \]

\[ G_3 = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \]

\[ \nu(G_3) = \frac{\frac{1}{2}}{\frac{3}{2}} = \frac{1}{3} \]

We can guess \( \nu(G_n) = \frac{1}{n} \)

Then \( G_{n+1} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{n} \end{pmatrix} \)
No saddle point

\[ v(G_{n+1}) = \frac{1}{n} \frac{1}{n+1} = \frac{1}{n+1} \]

By induction,

\[ v(G_n) = \frac{1}{n}. \]

For both players optimal

\[ \left( \frac{c}{n}, \frac{n-1}{n} \right) \]