Mar 17

Multistage game

Recursive game
Example: Guess it

From a deck of \( m+n+1 \) distinct cards,

- \( m \) cards \( \rightarrow \) Player I
- \( n \) cards \( \rightarrow \) Player II

1 \( \rightarrow \) target card

is placed face down on the table.

Players know their own cards
but not the rest.
Objective: guess the target card.

I starts. Each move:
1) guess the card. The game ends if correct win or incorrect lose.
2) ask if the other player holds a certain card.
   If the other player has the card, the card is removed from the play.
Let's say the game is $A_{m,n}$

$n = 0 \Rightarrow A_{m,0}$ is easy

I win.

$m = 0 \Rightarrow A_{0,n}$

I has $\frac{1}{n+1}$ chance to win

otherwise II wins.

How about $A_{m,n}$?

I guess $\frac{1}{n+1}$ chance to win.

or ask if II has card $A$. 
Should I always choose A that is not his card?

Does I choose from \( n + m + 1 \) card or from \( n + 1 \) card?

If from \( n + 1 \) card,

There is \( \frac{n}{n+1} \) chance A is in II's hand.

Remove a card.

\[ \Rightarrow \text{ C}_{n+1, m} \text{ for II.} \]
We denote this by $\overline{A_{n-1,m}}$.

There is $\frac{1}{n+1}$ that $A$ is the target card.

In this case, if II knows I is always honest, II wins.

If $A$ is chosen from $1+n+m$ cards. Then if $A$ is NOT in II's hand, II's situation is not improved.
So choose A from n+m+1 cards.

If A is from I's hand
We call it bluff.

If A is not from I's hand
We call it honest.
So bluff is necessary.

Bluff and honest are really different strategies.
Bluff: No card is removed when I guess. I can choose to call A → lose. I wins.

I can ignore (she thinks I is bluffing).

The game becomes I card removed from 2, II play next → \( A_{n,m-1} \)

honest: \( \frac{n}{n+1} \) chance a card is removed

For the rest \( \frac{1}{n+1} \)
If II thinks I is honest
→ II wins.
If II ignores, I would know
the card is the target card
→ I wins.
So the point is I has
3 different strategies
\{ honest, bluff, guess \}
\[ \text{If} \quad \text{win} = 1 \]
\[ \text{lose} = 0 \].
ignore \[ \frac{n}{n+1} \bar{A}_{n-1,m} + \frac{1}{n+1} \bar{A}_{n+1,m} \]

bluff \[ \bar{A}_{n,m-1} \]

guess \[ \frac{1}{n+1} \]

Actually the last row is dominated if \( m \geq 1 \) \( n \geq 1 \).

If at the very beginning I ask for a card with equal probability \( \frac{1}{n+1+m} \)
\[
\left( \frac{m}{n+1+m} \right) \text{ row } 1 + \frac{1}{n+1+m} \text{ row } 2
\]

Then II either guess or ignore.

If II guesses \( \frac{1}{n+1} \) chance win.

\[
\to \quad \frac{m}{n+1} \geq \frac{1}{2} \geq \frac{1}{n+1}
\]

If II ignores.

Then II guesses \( \frac{1}{n+1} \) or \( \frac{1}{n} \)
\[
\Gamma_{mn} = \left( \frac{n}{n+1} \Gamma_{n-1,m} + \frac{1}{n+1} \frac{n}{n+1} \Gamma_{n-2,m} \right)
\]

\[
\nu(\Gamma_{n,m}) = \nu_{mn}
\]

\[
\nu(\Gamma_{n,m}) = 1 - \nu_{mn}
\]

There is no saddle point since bluff is necessary
\[ V_{n,m} = \left( 1 - V_{n,m+1} \right) \]

\[
\left[ n \left( 1 - V_{n-1,m} \right) + 1 \right] - n \left( 1 - V_{n-1,m} \right) \wedge \\
\frac{n \left( 1 - V_{n-1,m} \right) + n + 2 - n \left( 1 - V_{n-1,m} \right)}{n \left( 1 - V_{n-1,m} \right) + (n+1) \left( 1 - V_{n,m+1} \right)}
\]

Get \( V_{n,m} \) recursively

\[ V_{n,m} = \frac{1 + n \left( 1 - V_{n-1,m} \right) \left( V_{n,m-1} \right)}{1 + (n+1) \left( V_{n,m-1} \right)} \]
What if \( A \) needs to recall \( A \) itself?

\[ A = (A!) \]

This is a recursive game.
It's NOT a finite game.
It's possible to repeat forever.
Assign a playoff $Q$ if the game is played forever.

If $Q \geq 1$, I can choose Row 1 and guarantee the playoff $1. \ II \rightarrow \text{column 2}$ assure loss 1.

If $Q < 1$, I has no optimal strategy.
But I can have a strategy, with average gain as close to 1 as possible. I can guarantee average payoff \( \geq 1 - \varepsilon \).

Such a strategy, guarantee a player payoff within \( \varepsilon \) of the value, is called \( \varepsilon \)-optimal.
For $1 \ (3, 3, 3)$

Since $3 \rightarrow 1$ will eventually choose Row 2, and the game stops.

$\Rightarrow$ I should choose Column 2 and stick with it.

$\rightarrow$ Payoff

\[
(1) \cdot 1 + 0 \cdot 3 = 1 - 1 = 3
\]
\[ \mathbf{C} = \left( \begin{array}{cc} A & 5 \\ 1 & 0 \end{array} \right), \quad Q \geq 1. \]

I \rightarrow \text{Row 1}.

If \( 1 \leq Q \leq 5 \)

II \rightarrow \text{Column 1}.

\[ \rightarrow v = Q \]

If \( Q > 5 \).

II \rightarrow \text{Column 2}

\[ v = 5. \]

If \( Q < 1 \).

I \text{ wants to stop. Stop on Row 2.} \]
II column 1. $V = 1$

Can we replace $a$ by $v$? the value?

$V = \text{Val}(1 0)

If $1 \leq v \leq 5$

$\implies$ $v$ a saddle point

$V = \text{Val}(1 0)$

So the value is always a solution of

$V = \text{Val}(1 0)$
But the equation
\[ v = \text{Val}(v, 5) \]
might have more solutions (solutions that are not the value)

Fact: This is always true for a recursive game
We can always replace \( a \) by the value \( v \).
and try to solve the equation

If there are finitely many solutions, find a way to exclude wrong answers.

Example:

\[ G = \begin{pmatrix} a & 0 \\ 0 & G \end{pmatrix}, \quad Q \]

\[ v = \text{val} \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix} \]

\[ v \geq 0, \quad \det = -1 - v^3 \neq 0 \]
$$A^{-1} = \frac{1}{-1-\sqrt{3}} \begin{pmatrix} -\sqrt{3} & -1 & \sqrt{3} \\ -1 & \sqrt{3} & -\sqrt{3} \\ \sqrt{3} & -\sqrt{3} & -1 \end{pmatrix}$$

$$V = (-1-\sqrt{3}) \cdot \frac{1}{-3V^2+3V-3}$$

$$= \frac{(v+1)(V^2-V+1)}{3(V^2-V+1)}$$

$$= \frac{V+1}{3}$$

$$3V = V+1$$

$$V = \frac{1}{2}$$
Example

\[ G_1 = \begin{pmatrix} G_2 & 0 \\ 0 & G_3 \end{pmatrix} \]

\[ G_2 = \begin{pmatrix} G_1 & 1 \\ 1 & 0 \end{pmatrix} \]

\[ G_3 = \begin{pmatrix} G_1^2 & 0 \\ 0 & 2 \end{pmatrix} \] by \( \frac{1}{2} G_1^2 \)

\[ v_1, v_2, v_3 \quad v_1 > 0, v_2 > 0, v_3 > 0 \] mixed

\[
\begin{cases}
  v_1 = \frac{v_2 v_3}{v_2 + v_3} \\
  v_2 = \frac{1}{2 - v_1} \\
  v_3 = \frac{4}{4 - v_1}
\end{cases}
\]
\[ 5v_1^2 - 12v_1 + 4 = 0 \]

\[ v_1 = 2 \text{ or } v_1 = \frac{2}{5}. \]

Now \( v_2 \leq 1 \) \( v_3 \leq 2 \)

(if stop) \( \pi (\frac{1}{2}, \frac{1}{2}) \)

\[ \Rightarrow \quad v_1 \leq 1 \]

\[ \Rightarrow \quad v_1 = \frac{2}{5} \]

\[ v_2 = \frac{5}{8} \]

\[ v_3 = \frac{10}{18} = \frac{10}{9} \]

\[ \Rightarrow \quad C_1 \quad \text{opt} \quad (\frac{16}{25}, \frac{9}{25}) \]

\[ C_2 \quad \left( \frac{5}{8}, \frac{3}{8} \right) \]

\[ C_3 \quad \left( \frac{5}{9}, \frac{4}{9} \right) \]