Mar 3.

The Extensive form of a game

Two person zero sum game

Extensive form encodes more information
Recall the game tree.

A directed graph \((T, F)\)

\[ T = \{ \text{vertices} \} \]

\(F\) is a function from \(T\) to the set of subsets of \(T\)

\[ \forall x \in T \]

\(F(x) = \text{the set of followers of } x \)
Each vertex represents a position of a game. The followers \( F(x) \) of position \( x \) are those positions that can be reached from \( x \) by one move.

Recall a path from a vertex \( x_0 \) to a vertex \( x_n \) is a sequence \( x_0, x_1, \ldots, x_n \) such that \( x_i \in E F(x_{i-1}) \) for \( i = 1, \ldots, n \).
For the extensive forms, we need

Definition, A **tree** is a directed graph, \((T,F)\) in which there is a special vertex \(r\) (called the root) such that for every other vertex \(t \in T\), there is a unique path from \(r\) to \(t\).
to initial position $t$ such that $F(t) = \emptyset$ terminal position.

Specify payoffs at each terminal.

For non-terminal positions

$\Rightarrow$ 3 groups

Some vertices assigned to Player II
Some vertices assigned to Player II
Some are chance nodes.
Chance nodes → rolling of a dice / dealing of poker cards ...

Basic End Game in Poker
Both players put $1, called the ante, on the table.
The money on the table is called the pot.
Player 1 is dealt a card.
 Winning card  Probability 1/4
  losing card  Probability 3/4.

Player 1 sees the card
 Player II doesn't.

Player 1 < check

If he checks, the card is inspected.
 Winning card, I win the pot
  losing card, II wins the pot.
If I bets, Player 1 puts 2 dollars more into the pot.

The II < fold
call

If she folds, loses the pot
If she calls, add 2 dollars more into the pot.

Winning card the pot → I
Losing card the pot → II
One problem here is that II doesn't know whether the card is winning or losing so the two vertices (II) should not which possible two
positions she is at.

If we don’t tell the difference between some positions, encircle them together

N

winning

\frac{1}{4}

losing \ \frac{3}{4}

bet

check

II

call

fold

1

bet

check

-1

call

fold

3

1

-3

1
A circle → an information set.
Such a graph is called a Kuhn tree.

Definition A finite two-person zero-sum game in extensive form is given by

1) a finite tree with vertices $T$
2) a payoff function that assigns a real number to each terminal vertex
3) a set of non-terminal vertices (representing chance nodes) and for each \( t \in T_0 \), a probability distribution on the edges leading from \( t \).

4) a partition of the rest of vertices (not in \( T_0 \), not terminal) into two groups of information sets

\( T_{11}, T_{12}, \ldots, T_{1k_1} \) (for Player I)

and \( T_{21}, T_{22}, \ldots, T_{2k_2} \) (for Player II)
5) for each information set $T_{jk}$ for player $j$, a set of labels $L_{jk}$, and for each $t \in T_{jk}$, a one-to-one mapping of $L_{jk}$ onto the set of edges leading from $t$.

Remark 5) means for $t_i, t_j \in T_{jk}$ from the same information set $t$, and $t_i$ and $t_j$ have the same moves, $L_{jk}$. 
The information set can be used to indicate a lack of information like the Basic Endgame.

It may also describe situations in which players forget previous moves.

\[
\begin{array}{c}
\text{I} \\
\text{II} \\
\end{array}
\]

\[
\begin{array}{c}
a \\
b \\
c \\
d \\
e \\
f \\
g \\
\end{array}
\]

\[
\begin{array}{c}
-1 \\
0 \\
1 \\
\end{array}
\]

\[
\begin{array}{c}
-1 \\
0 \\
1 \\
\end{array}
\]

\[
\begin{array}{c}
\text{forget the previous c vice f,g.}
\end{array}
\]
From strategic form
to an Extensive form

Given a strategic form
\((X, Y, A)\)

\(\rightarrow\) an extensive form.

A strategic form

Two players make choices simultaneously

\(\rightarrow\) Player I make moves first, then Player II make moves.
Since II does not know the choices of I.

Add information set.
Reduction of an Extensive form to a strategic form.

A pure strategy =

For each of the information set, a rule to make choices.
For Player $I$, information sets $T_{11}, T_{12}, \ldots, T_{1k_1}$

$\text{set of moves}$

$L_{11}, L_{12}, \ldots, L_{1k_1}$

$\cup$

$x_1, x_2, \ldots, x_{k_1}$

A pure strategy for $I$

is a $k_1$-tuple $(x_1, x_2, \ldots, x_{k_1})$

where $x_i \in L_{1i}$ for each $i$

$X = \{ (x_1, \ldots, x_{k_1}) \mid x_i \in L_{1i} \}$
Similarly

\[ \text{T}_{21} \text{ Take information set for II.} \]

\[ Y = \{ (y_1, \cdots, y_{k2}) \mid y_j \in \text{L}_{2j} \} \]

If there is no chance node

\[ A(x, y) \text{ is the payoff} \]

If there are chance nodes

\[ A(x, y) \text{ is the average value} \]
Example the basic Endgame.

\[ b = \text{bet} \quad c = \text{check} \]

\[ X = \{ (b, c), (b, b), (c, b), (c, c) \} \]

\[ Y = \{ c, f \} \]

\[ c = \text{call} \quad f = \text{fold} \]

\[ A ((b, b), c) = \frac{1}{4} \times 3 + \frac{3}{4} (-3) \]
\[ = -\frac{3}{2} \]
\[(b, b) \quad \begin{pmatrix} \frac{3}{2} & 1 \\ 0 & -\frac{1}{2} \end{pmatrix} \]

\[(b, c) \quad \begin{pmatrix} -2 & 1 \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} \] dominated

\[(c, b) \quad \begin{pmatrix} \frac{3}{2} & 1 \\ 0 & -\frac{1}{2} \end{pmatrix} \]

\[(c, c) \quad \begin{pmatrix} -2 & 1 \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} \]

No saddle point

\[-\frac{3}{2} p + 0 = p - \frac{1}{2} (1-p)\]

\[p = \frac{1}{6}.\]

\[-\frac{3}{2} q + (1-q) = -\frac{1}{2} (1-q)\]

\[q = \frac{1}{2}.\]
\[ \vec{p} = \left( \frac{1}{6}, \frac{5}{6}, 0, \infty \right) \]
\[ \vec{\tau} = \left( \frac{1}{2}, \frac{1}{2} \right) \]
\[ V = -\frac{1}{4}. \]