Mar 31

Noncooperative Game

Strategic Equilibrium
\[
\begin{pmatrix}
(2,0) & (1,3) \\
(0,1) & (3,2)
\end{pmatrix}
\]

We study this position.

If Player I changes his strategy, while II is holding Column 2.

The payoff 3 \rightarrow 1.

If Player II changes her strategy, while I
is holding Row 2,

The payoff for II 2 \to 1

So (3,2) is a position

where if a player thinks

that the other player

is not going to change

his strategy, then he should

not change his strategy either.

This is called a strategic
equilibrium
The condition

\[(3, 2)\]

\[3 = u_1(2, 2)\]
\[2 = u_2(2, 2).\]

\[u_1(2, 2) \geq u_1(1, 2)\]

If I changes to Row 1

and \[u_2(2, 2) \geq u_2(2, 1)\]

If II changes to Column 1
In other words for (2,2)

This Row 2 is the best response for II against II's Column 2.

Column 2 is the best response for II against I's Row 2.
A finite n-person game. The sets of pure strategies are $X_1, \ldots, X_n$.

The payoff functions are $u_1, u_2, \ldots, u_n$.

Each $u_i$ is a function on $X_1 \times X_2 \times \cdots \times X_n$.

$x_1 \in X_1, x_2 \in X_2, \ldots, x_n \in X_n$

$u_i(x_1, x_2, \ldots, x_n)$ is the payoff for $i$ if strategies
\[ x_1, x_2, \ldots, x_n \text{ are chosen.} \]

**Definition.** A vector of pure strategies \((x_1, x_2, \ldots, x_n)\) with \(x_i \in X_i\) (for \(i = 1, \ldots, n\)) is said to be a **pure strategic equilibrium (PSE)** if for all \(i = 1, 2, \ldots, n\), and for all \(x \in X_i\)

\[
\forall i (x_1, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) \Rightarrow \forall i (x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_n)
\]
Here 1 means
if i changes strategy from
xi to x, while the
other players all keep
their indicated strategies,
the payoff can not get
better. In other words,
xi is the best response
against (x1, ..., xi-1, -) xi+1
... xn)
A particular selection of strategy choices of the players forms a PSE if each player is using a best response to the strategy choices of the other players.
Both (3,3) (5,5) PSE.

You only need to compare

Note that in (3,3)
Both players can change their strategies and get better payoffs (5,5)
But they can not get
better payoff if only one of them changes strategy.

\[
\begin{pmatrix}
(3,3) & (4,3) \\
(3,4) & (5,5)
\end{pmatrix}
\]

Again (3,3) (5,5) are PSE.

But in (3,3) No one is hurt by changing strategies.

and if both change

\[\rightarrow (5,5)\] better.
Mixed strategies

If k strategies

The set of mixed strategies is

\[ P_k = \{ (p_1, \ldots, p_k) : p_i \geq 0 \quad i = 1, \ldots, k \} \]

\[ \sum_i p_i = 1 \]

For example, \( X_i \) has mi pure strategies

\[ X_i^* = P m_i \]

The set of mixed strategies for Player i
If each player chooses a mixed strategy $\vec{\pi} \in X_i^*$

$$\vec{\pi}_i = (\pi_{i1}, \pi_{i2}, \ldots \pi_{imi})$$

Remember $i$ is the index for player $i$

$m_i = \# \text{ of pure strategies}$

The average payoff is

$$g_i (\vec{\pi}_1, \ldots \vec{\pi}_n) = \sum_{i_1=1}^{m_1} \cdots \sum_{i_n=1}^{m_n} \pi_{i_1} \cdots \pi_{i_n} u_j (i_1, \ldots i_n)$$
If you get confused think about $n = 2$

$p_1 = (p_1, p_2, \ldots, p_n)$

$q_2 = (q_1, q_2, \ldots, q_n)$

$g_j(p_1, p_2) =$

$\begin{pmatrix}
p_1 & \cdots & p_n
\end{pmatrix}
\begin{pmatrix}

u_{j(i,k)}
\end{pmatrix}_{m \times n}
\begin{pmatrix}
q_1 \\
\vdots \\
q_n
\end{pmatrix}$

matrix for Player $j$. 

Definition

A vector of mixed strategy choices \((\vec{P}_1, \vec{P}_2, \ldots, \vec{P}_n)\) with \(\vec{P}_i \in X_i^*\) for \(i = 1, \ldots, n\) is said to be a strategic equilibrium (\(SE\)) if for all \(i = 1, 2, \ldots, n\), and for all \(\vec{p} \in X_i^*\),

\[
g_i(\vec{P}_1, \ldots, \widehat{\vec{P}_i}, \ldots, \vec{P}_n) \\
\geq g_i(\vec{P}_1, \ldots, \widehat{\vec{P}_i}, \vec{p}, \widehat{\vec{P}_i}, \ldots, \vec{P}_n)
\]

Change \(\vec{P}_i\) to \(\vec{p}\).
Again \( (\overline{p}_1, \overline{p}_2, \ldots, \overline{p}_n) \)

is a SE means

Each \( \overline{p}_i \) for all \( i=1, \ldots, n \)

is a best response

against the other indicated mixed strategies

Why is SE important?

It always exists!
Theorem (John Nash)

Every finite n-person game in strategic form has at least one strategic equilibrium.

This theorem is worth a Nobel Prize.

See the movie

A beautiful Mind

There might be many SE in a game
Example

\[
\begin{pmatrix}
(3, 3) & (0, 2) \\
(2, 1) & (5, 0)
\end{pmatrix}
\]

Like saddle point PSE

find the best response in each row

find the best response in each column.

If some position have two circles, done!
(3, 3) (5, 5) are both PSE

How about SE that is not PSE?

Equalizing strategies (Both have)

\[ \rightarrow \] SE

\[ \{ (p, 1-p), (q, 1-q) \} \]

if

\[ U_2((p, 1-p), 1) = U_2((p, 1-p), 2) \]

then it doesn't benefit II to change her mixed strategy.

All mixed strategies are the same.
against \((p, 1-p)\)

In particular \((q, 1-q)\) is a best response against \((p, 1-p)\)

\[
H \quad u_1 \left(1, \left(q, (1-q)\right)\right) = u_1 \left(2, \left(q, (1-q)\right)\right)
\]

then any mixed strategy is a best strategy against \((q, 1-q)\).

So in particular \((p, 1-p)\) is a best response against \((q, 1-q)\).
$(p, 1-p)$ makes II indifferent to the payoffs $\beta$

$(q, 1-\beta)$ makes I indifferent to the payoffs $A$

$$3p + (1-p) = 2p + 5(1-p)$$

$$p = \frac{4}{5}$$

$$3q + 0 = 2q + 5(1-q)$$

$$q = \frac{5}{6}$$

$$(\frac{4}{5}, \frac{1}{6}), (\frac{5}{6}, \frac{1}{6}) \rightarrow SE$$

Payoffs $\left(\frac{5}{2}, \frac{13}{5}\right) = (V_I, V_{II})$
Example 2.

Judy Hopps and Nick Wilde go to cinema. They prefer different movies (NOT Zootopia). But going together is preferable to going alone. The preference is the table.
\[
J(a, b) = \begin{pmatrix}
(2, 1) & (0, 0) \\
(0, 0) & (1, 2)
\end{pmatrix}
\]

So, \((a, a)\) and \((b, b)\) are PSE.

SE?

\[
2q + 0 = 0 + (1 - q)
\]

\[
q = \frac{1}{3}
\]

\[
p + 0 = 0 + 2(1 - p)
\]

\[
p = \frac{2}{3}
\]

\[
\left(\frac{2}{3}, \frac{1}{5}\right), \left(\frac{1}{3}, \frac{2}{3}\right) \rightarrow SE
\]

But average payoff.
\[
\begin{pmatrix}
\frac{2}{3} & \frac{1}{3} \\
\frac{2}{3} & \frac{1}{3}
\end{pmatrix}
\begin{pmatrix}
2 & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
\frac{1}{3} \\
\frac{2}{3}
\end{pmatrix}
\]

\[=rac{2}{3}\]

\[
\begin{pmatrix}
\frac{2}{3} & \frac{1}{3} \\
\frac{2}{3} & \frac{1}{3}
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & 2
\end{pmatrix}
\begin{pmatrix}
\frac{1}{3} \\
\frac{2}{3}
\end{pmatrix}
\]

\[=rac{2}{3}\]

Payoff
\[
\begin{pmatrix}
\frac{2}{3} \\
\frac{2}{3}
\end{pmatrix}
\]

Worse than both PSE.
Example The Prisoner's Lemma

Two thieves are caught.
An attorney offers them deals separately.

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>If confess, the other silent</td>
<td>free</td>
</tr>
<tr>
<td></td>
<td>4 years</td>
</tr>
<tr>
<td>If both confess</td>
<td>both 3 years</td>
</tr>
<tr>
<td>If both silent</td>
<td>both 1 year</td>
</tr>
</tbody>
</table>
Now free $\rightarrow$ Payoff 4.
4 years $\rightarrow$ Payoff 0.

\[
\begin{bmatrix}
  s & c \\
  (3, 3) & (0, 4) \\
  (4, 0) & (1, 1)
\end{bmatrix}
\]

PSE (1, 1)

If they are NOT allowed to communicate, or they don't trust each other
$\rightarrow$ (1, 1)
For I, \( s \) is dominated by \( c \).

For II, \( s \) is dominated by \( c \).

\[ \rightarrow (c, c) \quad (1, 1) \]

However, \( (3, 3) \) is a better position for both.