Notes on Examples of SU(2),SO(3)-actions on Positively Curved 6-manifolds

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In this article we want to list a few examples of isometric SU(2),SO(3)-actions on all known examples of positively curved 6-manifolds, namely $S^6$, $\mathbb{CP}^3$, $SU(3)/T^2$, $SU(3)/\!/T^2$. We list the isotropy groups and give geometric description of the orbit spaces.

1. **SO(3)-action on $S^6$,** given by $A(\vec{x}, \vec{y}, z) = (A\vec{x}, A\vec{y}, z)$, $A \in SO(3)$, $\vec{x}, \vec{y} \in \mathbb{R}^3$, $z \in \mathbb{R}$, $(\vec{x}, \vec{y}, z) \in S^6$. This action has 2 isolated fixed points $(0, 0, \pm 1)$. The orbit space is a 3-ball, where the interior points correspond to principal orbits with trivial isotropy, the boundary 2-sphere minus 2 points corresponds to singular orbits with SO(2) isotropy, and 2 antipodal points correspond to 2 fixed points. The singular orbits consist of points whose $\vec{x}$, $\vec{y}$-components are linearly dependent but not both zero. Since the orbit space has boundary, we may apply the Soul Theorem.

2. **SO(3)-action on $\mathbb{CP}^3$,** given by $A(z_1 : z_2 : z_3 : z_4) = (A(z_1 : z_2 : z_3)^T : z_4)$, $A \in SO(3)$, $(z_1 : z_2 : z_3 : z_4) \in \mathbb{CP}^3$. This action has a unique isolated fixed point $(0 : 0 : 0 : 1)$. All the isotropy types (and the corresponding points) are:

   1. $SO(3)$: isolated fixed point $(0 : 0 : 0 : 1)$;
   2. $SO(2)$: ($\vec{x} \in \mathbb{R}^3 : w \neq 0 \in \mathbb{C}$);
   3. $O(2)$: ($\vec{x} \in \mathbb{R}^3 : 0$), this is a unique orbit;
   4. $SO(2)$: ($\vec{v} : 0$), where $\vec{v} \in \mathbb{CP}^2$ satisfies $(v, v) = \sum_{i=1}^{3} v_i^2 = 0$, this is also a unique orbit;
   5. $Z/2$: ($\vec{v} : 0$), where $\vec{v} \in \mathbb{CP}^2 \setminus \mathbb{RP}^2$ satisfies $(v, v) = \sum_{i=1}^{3} v_i^2 \neq 0$;
   6. $id$: ($\vec{v} \in \mathbb{C}^3 \setminus \mathbb{R}^3 : w \neq 0 \in \mathbb{C}$).

   The orbit space in this case is a 3-ball, where interior points minus a line segment correspond to principal orbits $id$, the boundary 2-sphere minus 2 points correspond to $SO(2)$-singular orbits (type 2), 2 points on the boundary correspond to $SO(3)$-orbit (fixed point) and $O(2)$-orbit respectively, and a line segment in the interior corresponds to $Z/2$-orbits connecting $O(2)$-orbit and $SO(2)$-orbit (type 4), and lastly a unique point in the interior corresponds to $SO(2)$-orbit (type 4).

For curiosity I applied Soul Theorem to the orbit space under Fubini-Study metric on $\mathbb{CP}^3$. According to my computation, the soul orbit, which is the orbit of maximal distance to the boundary 2-sphere, must consist of points whose last coordinate is 0, thus must be of type 3, 4, or 5. I suspect it is of type 4, i.e. the unique $SO(2)$-orbit in the interior, but I didn’t finish my computation. In order to compute the distance from an interior point to the boundary, I used Lagrange multiplier, but the equations seem very complicated.
3. Another $SO(3)$-action on $\mathbb{C}P^3$, induced from one $SU(2)$-action. Let $A \in SU(2)$ act on $\mathbb{C}P^3$ via $A(\vec{x}, \vec{y}) = (Ax, Ay)$, $\vec{x}, \vec{y} \in \mathbb{C}^2$. This action is ineffective since $-Id \in SU(2)$ acts trivially, thus descends to an $SO(3)$-action. The orbit space is a 3-ball, where interior points correspond to principal orbits $id$, and the boundary points correspond to singular $U(1)$-orbits. The singular orbits consist of points whose $\vec{x}$, $\vec{y}$-components are linearly dependent.

4. Another $SO(3)$-action on $\mathbb{C}P^3$, coming from the 4-dim complex irrep of $SU(2)$. The irreducible action of $SU(2)$ on $\mathbb{C}^4$ induces one on $\mathbb{C}P^3$, which is ineffective with kernel $\mathbb{Z}/2$ and descends to $SO(3)$. We realize $\mathbb{C}^4 = \text{span}_\mathbb{C}\{x^3, x^2y, xy^2, y^3\}$ and write $(a, b, c, d) = ax^3 + bx^2y + cxy^2 + dy^3$. Under this identification, $\begin{bmatrix} a & b \\ -b & a \end{bmatrix} \in SU(2)$, $|a|^2 + |b|^2 = 1$ corresponds to

$$\begin{bmatrix} a^3 & -a^2b \\ 3ab^2 & a(3|a|^2 - 2) \\ 3a^2b & b(3|a|^2 - 1) \end{bmatrix} \begin{bmatrix} -a^2b \\ 3ab^2 \\ b^3 \end{bmatrix},$$

$$\begin{bmatrix} \frac{a^3}{3a^2b} & \frac{-a^2b}{3ab^2} \\ \frac{a(3|a|^2 - 2)}{b(3|a|^2 - 1)} & \frac{-a^2b}{b^3} \end{bmatrix}. \tag{1}$$

The orbit space is a 3-sphere, and the orbit types are as follows:

1. the principal isotropy is trivial.

2. 2 singular orbits with $U(1)$-isotropy, corresponding to $x^3$, $x^2y$ respectively. The slice representation of $U(1)$ at the singular orbit $x^3$ has slope $(2, 3)$ and the slice rep at $x^2y$ has slope $(1, 2)$, which matches up with exceptional orbits.

3. an exceptional orbit with isotropy $S_3$, with representative $(1, 0, 0, 1) \sim (1, 0, 3, 0)$. This $S_3$ is the image of the binary dihedral group of order 12 generated by $\begin{bmatrix} \zeta_6 & 0 \\ 0 & \zeta_6 \end{bmatrix}$, $\begin{bmatrix} 0 & \zeta_{12} \\ -\zeta_{12} & 0 \end{bmatrix} \in SU(2)$. The slice is the real linear span of $(1, 0, 0, -1), (0, 1, 1, 0), (0, i, -i, 0)$, where $S_3$ acts on $(1, 0, 0, -1)$ as reflection and acts as the dihedral group on the span of the other 2 vectors. One axis of reflection in the dihedral group is $(0, (1 - \sqrt{3}i), (1 + \sqrt{3}i), 0)$.

4. a 1-parameter families of exceptional orbits with isotropy $\mathbb{Z}/2$ with representatives $(1, 0, t, 0)$ where $t \in \mathbb{R}_{>0}$, $t \neq 3$, whose orbit stratum consists of a path connecting the singular $x^3$ to the exceptional orbit 3 and another path connecting 3 to $x^2y$. Note: $(1, 0, t, 0)$ lies in the same orbit as $(0, 1, 0, t)$, and $(0, 1, 1, 0)$ lies in the orbit of $(1, 0, 1, 0)$.

5. a 1-parameter families of exceptional orbits with isotropy $\mathbb{Z}/3$ with representatives $(1, 0, 0, \mu) \sim (1, 0, 0, \mu^{-1})$, $0 < |\mu| < 1$. The orbit stratum is a path connecting the singular orbit $x^3$ and the exceptional orbit 3 $(1, 0, 0, 1)$.

**Remark 1** I computed the eigenvalues and eigenvectors of the Lie algebra action to determine singular orbits, and by diagonalizing $SU(2)$ matrices we know that exceptional isotropy near singular orbits must consist of matrices with "special" eigenvalues, i.e. 4-th or 6-th root of unity.
5. SU(2)-action on $S^6$ given by $A(\vec{x}, \vec{y}) = (A\vec{x}, A\vec{y})$, $A \in SU(2)$, $\vec{x} \in \mathbb{R}^4$, $\vec{y} \in \mathbb{R}^3$, $(\vec{x}, \vec{y}) \in S^6$. The action on the $\vec{x}$-component comes from the real 4-dim irrep of SU(2), i.e. the realification of the standard SU(2)-action on $\mathbb{C}^2$, and the action on $\vec{y}$-component comes from the standard SO(3)-action on $\mathbb{R}^3$. The orbit space is the 3-sphere, with only one singular orbit corresponding to the points whose $\vec{x}$-component vanishes, and the singular isotropy is $U(1)$. The principal isotropy is trivial.

Remark 2 In this case, if we are given the orbit structure, then by van Kamper theorem we know the G-space is simply connected, and by Mayer-Vietoris sequence we know it’s a homology sphere. Thus by the solution to Poincare conjecture we know the G-space must be homeomorphic to the 6-sphere.

6. SU(2)-action on the homogeneous flag manifold $SU(3)/T^2$, given by left multiplication. The orbit space is a 3-sphere with 3 singular isotropy corresponding to the matrices

$$Id, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$ 

The principal isotropy is trivial.

7. SU(2)-action on the biquotient $SU(3)//T^2 = (z, w, zw) \setminus SU(3)/(1, 1, z^2w^2)^{-1}$, $z, w \in S^1$. SU(2) acts from the right as the first 2 block of SU(3), commuting with $T^2$-action. The orbit space is again a 3-sphere, with trivial principal isotropy and 3 singular $U(1)$-orbits with representatives

$$[Id] = \begin{bmatrix} \begin{pmatrix} z \\ 0 \\ 0 \end{pmatrix} A & 0 \\ 0 & w_{a1} & w_{a2} \\ 0 & zw_{a1} & zw_{a2} \end{pmatrix}, \begin{bmatrix} \begin{pmatrix} \ast \\ \ast \\ \ast \end{pmatrix} & 0 \\ 0 & 0 & \ast \\ 0 & 0 & \ast \end{bmatrix}. \end{bmatrix}.$$

And the interesting thing is that in this case we have a 1-parameter family of exceptional orbits with isotropy $\mathbb{Z}/3$, connecting the second and the third singular orbits above. The representatives of exceptional orbits are

$$\begin{bmatrix} c_1a_1 & c_1a_2 & f_1 \\ c_2a_1 & c_2a_2 & f_2 \\ b_1 & b_2 & 0 \end{bmatrix} \in SU(3).$$

Note: when $c_1$ (resp. $c_2$) becomes 0, we get back the second (resp. the third) singular orbit. The vectors $(a_1, a_2)$, $(b_1, b_2)$ are actually (left) eigenvectors of elements in the exceptional isotropy.

Remark 3 I computed the $U(1)$-fixed point set in this and the previous example. It is a set of 6 isolated points, namely 2 antipodal points from each singular orbit (homeomorphic to $S^2$). This matches up with the conclusion that the Euler characteristic of fixed point set under torus action is the same as that of the whole manifold.

Remark 4 If we are given the orbit structure, by Mayer-Vietoris sequence the G-space has the same Betti numbers as $SU(3)//T^2$, i.e. $b_0 = b_6 = 1$, $b_2 = b_4 = 2$, $b_{2i+1} = 0$. 
8. **SO(3)-action on SU(3)/T^2**, given by left multiplication. This action has orbit space a 3-sphere, with trivial principal isotropy. The principal orbits consist of matrices all of whose 3 columns are not proportional to real vectors. For simplicity, we say they are "complex" vectors and if a vector is proportional to a real vector, we just call it real vector. There are 3 singular orbits with SO(2)-isotropy, and one of them has representatives \([\vec{v}, \bar{\vec{v}}, \vec{w}] \in SU(3)\), where \(\vec{v} \in \mathbb{C}^3\), \(\vec{w} \in \mathbb{R}^3\) are column vectors, \(\bar{\vec{v}}\) is the complex conjugation of \(\vec{v}\), and \(\bar{\vec{v}}\) satisfies \((\vec{v}, \bar{\vec{v}}) = \sum_{i=1}^{3} v_i^2 = 0.\) The other two have similar representatives with \(\vec{w}\) put in the first and second column.

Exceptional orbits are also interesting in this example. There is one exceptional orbit with \(\mathbb{Z}/2 \oplus \mathbb{Z}/2\)-isotropy, corresponding to the \(SO(3)\) matrices in \(SU(3)\). Besides that, there are three 1-parameter families of \(\mathbb{Z}/2\)-orbits. Their representatives are like \([\vec{v}_1, \vec{v}_2, \vec{w} \in \mathbb{R}^3] \in SU(3)\), where \((\vec{v}_1, \vec{v}_1) \neq 0\) \(((\cdot, \cdot)\text{ is the complex linear extension of real inner product})\). Of course the column \(\vec{w}\) can be permuted around. In the orbit space, these three families of \(\mathbb{Z}/2\)-orbits are line segments connecting the three singular orbits with the \(\mathbb{Z}/2 \oplus \mathbb{Z}/2\)-orbit. The whole picture has a 3-fold symmetry, which might have something to do with the Weyl symmetry.