In what follows, you may use the result of a given part in a succeeding part of the problem, not the other way round.

a) Say T is a finite set and S is any (not necessarily proper) subset of T. Show that S is finite.

b) Assume T and W are both finite sets. Prove that $T \cup W$ is a finite set.

c) Now assume you know about $\mathbb{Z}$ and the results we proved about it in class. Thus, $\mathbb{Z}$ is infinite because S = all even integers is a proper subset and $\varphi: S \to \mathbb{Z}$ given by $\varphi(n) = \frac{1}{2}n$ is bar bijection. Now let $\mathbb{N}$ stand for the collection of all integers ≥ 1. If $k \in \mathbb{N}$, call S_k (a subset of $\mathbb{N}$) the segment up to $k \iff$ its complement in $\mathbb{N}$ (that is, all $n \in \mathbb{N}$ which are not in S_k) has these properties

① $k+1 \in$ complement & is its least element.

and ② If $n \in$ complement then automatically $n+1$ is also in the complement. (Let's denote the complement of S_k by T_k.)

Prove that S_k is finite (all fixed $k \in \mathbb{N}$) and T_k is infinite.

d) If $k, l \in \mathbb{N}$ & $k \neq l$, prove there is no bijection $\varphi: S_k \to S_l$.

e) Let T be any finite set. Prove that there is one and only one $k \in \mathbb{N}$ so that there is a bijection $S_k \to T$.

BIII) We consider $\mathrm{Aut}(\mathbb{R})$, the collection of all automorphisms of $\mathbb{R}$ - the real nos. Thus, $\varphi \in \mathrm{Aut}(\mathbb{R}) \iff$ ① φ is a bijection, $\mathbb{R} \to \mathbb{R}$ & ② φ is a ring homomorphism $\mathbb{R} \to \mathbb{R}$.

a) Explain exactly what such a φ does to the integers (which are inside $\mathbb{R}$). Give a formula for φ when it's restricted to operate only on integers.

b) Same as a) but now replace $\mathbb{Z}$ by $\mathbb{Q}$.

c) For this part you'll need two facts about $\mathbb{R}$, they were proved in Math 360.

Fact 1 Between any two non-equal real numbers, there is a rational number different from both.

Fact 2. If x is a positive real number, then x is the square of a positive real number.

Find explicitly all automorphisms of $\mathbb{R}$.

(Suggestion: Say $\varphi \in \mathrm{Aut}(\mathbb{R})$, show φ preserves order; that is, if $a > b$ then $\varphi(a) > \varphi(b)$.)