Mathematics 371 October 16, 2012 (Shatz)
Assignment #3 Due October 30, 2012

## Part A

54/1.6; 355/3.4, 3.7, 3.8, 3.11; 356/6.7, 357/8.1

## Part B

(BI) Do problem (BI) of the last HW (#2) over again except for part a). Instead of part a) do
part e) Find all solutions ($xyz \neq 0$) to
$$X^4+Y^4=Z^4 \text{ and } X^4+Y^4=Z^2.$$
Prove all your assertions.

BII) Here we study rings (not necc. commutative) which may or may not have unity elts. Let $R$ be such a ring.

a) We make a new ring, $R^\#$, from $R$ as follows: $R^\#$ consists of pairs, $(r, n)$, where $r\in R$ and $n\in \mathbb{Z}$. Addition is as usual, viz:
$$(r,m)+(s,n)=(r+s,\ m+n).$$
Mult. is different:
$$(r,m)*(s,n)=(rs+nr+ms,\ mn).$$
Of course, $nr$ means $r+r+\cdots+r$ ($n$-times) if $n\geq 0$ and is $-r+-r+\cdots+-r$ ($|n|$-times) if $n<0$.

If you're in doubt, check privately that $R^\#$ is a ring under these rules. Show that $R^\#$ always has a unity element – exactly which element is it?

b) Consider the map $\varphi: R \to R^\#$, given by $\varphi(r)=(r,0)$. Show (quickly) it's a homomorphism. What is the kernel of $\varphi$? Show $R^\#$ is commutative $\iff$ $R$ is commutative. Show further that if $R$ does possess a unity elt, $1_R$, then $\varphi(1_R)$ is NOT the unity of $R^\#$.

c) If $I$ is any ideal of $R$, prove $\varphi(I)$ is an ideal of $R^\#$ (two-sided if $I$ is two-sided, etc.). So, in particular, $\varphi(R)$ is an ideal of $R^\#$ & two-sided at that. Identify, up to isomorphism, the ring $R^\#/\varphi(R)$. Suppose every ideal (left, or right, or 2-sided as the assumption might be) of $R$ is finitely generated. Is the same statement true for $R^\#$? Is the converse valid? By the way, you should exhibit ideals, $J$, of $R^\#$ not of the form $\varphi(I)$ where $I$ is an ideal of $R$.

d) For which commutative rings (with unity), if any, is $R^\#$ a principal ideal ring? Is $R^\#$ ever an integral domain?

e) Assume $R$ is a comm. ring with unity. Let $J$ = the prin. ideal of $R^\#$ generated by the element $(-1_R, 1_{\mathbb{Z}})$. Describe the ring $R^\#/J$.

(over)

B III) Here, $R$ will be a comm. ring with unity (we know lots of these). Write $M_n(R)$ for the ring of $n \times n$ matrices with entries from $R$ — we are interested in 2-sided ideals, $J$, of $M_n(R)$.

a) Suppose each such $J$ is actually finitely generated. Need every ideal, $I$, of $R$, also be finitely generated? Prove all your assertions.

b) Conversely, if all ideals $I$, of $R$ are finitely generated, are all 2-sided ideals, $J$, of $M_n(R)$ also finitely generated? (Look at $n = 2$ first, try to find a way of relating such $J$'s to such $I$'s.)

c) Fix $n$. Suppose you know more: Every ideal of $R$ is gen'd by $\leq M$ elts. (E.g., $R$ is a prin. ideal ring, so $M = 1$.) Can you find an explicit fcn $\psi(M)$ so that each two sided ideal of $M_n(R)$ is gen'd by $\leq \psi(M)$ elts., or is this impossible?