Mathematics 371 Fall, 2012 (SHATZ)
Assignment #5 Due November 27, 2012

Part A

AI) 356/3.7 and 357/7.4
AII) 357/8.4
AIII) 358/M.2
AIV) 358/M.3
AV) 358/M.4

Part B

Before we do ~~part~~ BI) we need a little analysis (à la Math 360). Recall that if $\sum_{n=1}^{\infty} a_n$ is an infinite series and if $\sum_{n=1}^{\infty} |a_n|$ converges so does our original series. (The proof is easy: $0 \le |a_n| - a_n \le 2|a_n|$. By the comparison test the series

$$\sum_{n=1}^{\infty} (|a_n| - a_n)$$

converges. But then the series $\sum_{n=1}^{\infty} a_n$, being the difference $\sum |a_n| - \sum(|a_n| - a_n)$ of converging series, is itself convergent.)

We know $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots$, $|x| < 1$.

Integrate & get

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots, \quad |x|<1$$
$$= -x - x^2\left(\frac{1}{2} + \frac{x}{3} + \frac{x^2}{4} + \cdots\right)$$

If $0 < x < 1$, then $\frac{1}{2} + \frac{x}{3} + \frac{x^2}{4} + \cdots < 1 + x + x^2 \cdots$

∴

$$\log(1-x) = -x - Kx^2, \text{ where } K < \frac{1}{1-x}.$$

We define an infinite product $\prod_{n=1}^{\infty}(1-a_n)$ (here, $0 < a_n < 1$) to be

$$\lim_{N\to\infty} \prod_{n=1}^{N}(1-a_n) = P$$

and say the product converges if & only if P exists and $P \neq 0$. (If $P = 0$, the product diverges to 0; if P doesn't exist then the product simply diverges.) We prove

Theorem. If $0 < a_n < 1$, then $\prod_{n=1}^{\infty}(1-a_n)$ converges $\iff \sum_{n=1}^{\infty} a_n$ converges. (It diverges to 0 iff the inf. series diverges.)

Proof. Say $\sum a_n$ converges, then for large n, $a_n < \frac{1}{2}$ so $\frac{1}{1-a_n} < 2$ ∴

$$\log(1-a_n) = -a_n - K_n a_n^2, \quad K_n < 2.$$

As $a_n < 1$, we have $a_n^2 < a_n$ ∴ the comparison test shows that $\sum \log(1-a_n)$ converges. The latter is

$$\lim_{N\to\infty} \sum_{n=1}^{N} \log(1-a_n) = \lim_{N\to\infty} \log\left(\prod_{n=1}^{N}(1-a_n)\right)$$

and as log is continuous, we get $\log \lim_{N\to\infty} \prod_{n=1}^{N}(1-a_n)$ exists ∴ the product converges (it's not zero).

If now $\prod_{n=1}^{\infty}(1-a_n)$ converges then $\lim_{N\to\infty}\prod_{n=1}^{N}(1-a_n)$ exists. Apply log & use its continuity, we get

$$\lim_{N\to\infty} \sum_{n=1}^{N} \log(1-a_n) = \lim_{N\to\infty}\left(\log \prod_{n=1}^{N}(1-a_n)\right) \text{ exists.}$$

∴ $\sum \log(1-a_n)$ converges ∴ $\sum(-a_n - K_n a_n^2)$ converges so $\sum a_n + K_n a_n^2$ converges. But $a_n \le K_n a_n^2 + a_n$;

(OVER)

so, the comparison test shows $\sum a_n$ converges. QED

A last fact you may need is this: ~~If~~ $\sum |a_n|$ converges, then we may rearrange the terms in the series $\sum a_n$ anyway we like and the re-arranged series still converges and to the same sum.

BI) ~~If~~ p is a prime number and s is a real number ≥ 1, write $E(p;s) = \frac{1}{1-p^{-s}}$.

a) Show that $\prod E(p;s)$ (product taken
10 over all prime nos.) p ~~numbers~~ converges if $s>1$.
~~and~~
b) Prove that $\prod_p E(p;s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$, even
15 if $s=1$; the meaning of the latter is taken as equality ~~of~~ the limits as $s \to 1$ of both sides. (Hint- Consider the geometric series for

$$\frac{1}{1-\left(\frac{1}{p}\right)^s},$$

be careful with interchange of limits and use a fact about factorization in $\mathbb{Z}$, proved in class.)

c) ~~In~~ the product $\prod_{p \leq M} E(p;1)$, separate out
10 terms with p alone from those with $p^2, p^3, \ldots$. Use this and b) to show $\sum_p \frac{1}{p}$ diverges and that

the partial sums grow like $\log(\log M)$ (where $p \leq M$). Use the integral test to explain Euler's surmise that

$$\lim_{n\to\infty} \frac{p_n}{n \log n} = 1;$$

here, $p_n = n^{th}$ prime.

d) ~~If~~ Euler's surmise is correct, show that
5 $$\lim_{x\to\infty} \frac{\pi(x)}{\left(\frac{x}{\log x}\right)} = 1,$$
where $\pi(x) = \#$ of primes $\leq x$. This latter fact is indeed correct (proved in 1898, about 150 years after Euler's work).

15 BII) 358/9.13
15 BIII) 382/M.4
BIV) Read §12.5 (pp 376–378). Make a similar discussion for $\mathbb{Z}[\sqrt{2}]$, $\mathbb{Z}[\sqrt{-2}]$. ~~In particular:~~
5 a) What are units ($\mathbb{Z}[\sqrt{2}]$ is a ring whose units you've met before)?
10 b) Which integer primes remain prime?
7 c) Are either of these Euclidean Domains, ~~PIDs~~, UFD's?
3 d) Is there ~~a~~ an integer prime (or primes) which becomes (become) a square in the big ring?
5 e) ~~If~~ a prime is not b) or d), how does it factor in the big ring?