Research Statement

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1 Introduction

My research interests focus on the interaction of noncommutative geometry and representation theory. In more detail, my current research work centers around

- *Equivariant K-theory* of (noncompact) real semisimple Lie groups G acting on (complex) flag variety $\mathcal{B}$. For noncompact groups the equivariant K-theoy $\mathrm{K}_G(\mathcal{B})$ is defined (in [7]) to be the K-theory of the reduced crossed product C^*-algebra. I have obtained a proof of Baum-Connes conjecture with coefficient in $C(\mathcal{B})$. Now I am working on computing $\mathrm{K}_G(\mathcal{B})$ and understanding its relation to the representation theory of G.

- *Differential graded-category and geometric representation theory*. Together with my advisor J. Block, I am trying to find a suitable curved-dg algebra $\mathcal{A}^\bullet$ such that the dg-category $\mathcal{P}_\mathcal{A}$, consisting of cohesive $\mathcal{A}^\bullet$-modules, is quasi-equivalent to the category of admissible representations of G.

- *Covariant Weil algebras*. I introduced the classic and quantum covariant Weil algebras $W_\tau(\mathfrak{g})$ and $\mathcal{W}_\tau(\mathfrak{g})$ in [21]. The covariant Weil algebras are simultaneous generalizations of Weil algebras ([1]) and family algebras ([15]). $W_\tau(\mathfrak{g})$ and $\mathcal{W}_\tau(\mathfrak{g})$ are curved-dg algebras whose curvatures have elegant expressions. It is hoped that covariant Weil algebras can be applied to the construction of Mackey's analogue in [13].

- *Formality and the Kashiwara-Vergne problem*. Drinfeld associators give solutions to the generalized Kashiwara-Vergne problem ([4]) as well as stable formality maps on Hochschild cochains ([10] and [11]). I am working with V. Dolgushev to find solutions to the generalized Kashiwara-Vergne problem coming from each stable formality map, which is compatible with the above two maps.

In this research statement I will describe the background of research, the result I have got and the future plan of my work.

2 Equivariant K-theory of noncompact semisimple Lie group acting on flag varieties

Let G be a real semisimple Lie group acting on a topological space X. According to [7], the equivariant K-theory as the K-theory of the reduced crossed product C^*-algebra

$$\mathrm{K}_G(X) := \mathrm{K}(C^*_{\text{red}}(G; C_0(X))) \tag{1}$$

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has a useful connection to Baum-Connes conjecture. Be aware that this is not the same as Kasparov's definition in [14].

When $X = \text{pt}$, V. Lafforgue in [17] studied the relation between $\mathrm{K}_G(\text{pt})$ and the representation theory of G. My concern is mainly the case that $X = \mathcal{B}$ is the flag variety of $G_{\mathbb{C}}$, the complexification of G. First of all I have proved in [21] that the assemble map

$$\mu_{\text{red},\mathcal{B}} : \mathrm{KK}^G(S, \mathcal{B}) \longrightarrow \mathrm{K}_G(\mathcal{B}) \tag{2}$$

is an isomorphism, where $S = G/U$ and U is the maximal compact subgroup of G. Roughly speaking, this means that we have the isomorphism

$$\mathrm{K}_G(\mathcal{B}) \cong \mathrm{K}_U(\mathcal{B}) \tag{3}$$

up to a shift of dimension. The proof is inspired by the *Matsuki correspondence* in [18] and relies on a careful study of the G-orbits on $\mathcal{B}$.

This result is a proof of Baum-Connes conjecture with coefficient on flag varieties, which is not the end of the story, instead, it is the start point. In fact $\mathrm{K}_G(\mathcal{B})$ reveals rich information about the representation of G.

There are two problems I am currently working on. The first is to find explicitly the expression of $\mathrm{K}_G(\mathcal{B})$ for any real simple simple Lie group G. The strategy is that that we first study the G-orbits on $\mathcal{B}$: The real group G action on $\mathcal{B}$ is not transitive and $\mathcal{B}$ is a finite union of G orbits (for finiteness see [18])

$$\mathcal{B} = \bigcup_{i=1}^{n} \mathcal{O}_i. \tag{4}$$

The orbit structure of $\mathcal{B}$ plays an important role in representation theory, see [19]. In our context, I have shown in [21] that $\mathrm{K}_G(\mathcal{B})$ is a repeatedly extension of the $\mathrm{K}_G(\mathcal{O}_i)$ and each $\mathrm{K}_G(\mathcal{O}_i)$ is relatively easy to understand. Now I am working on:

Problem 1. *Compute $K_G(\mathcal{B})$ explicitly from $K_G(\mathcal{O}_i)$ and find its relation with representations of G. For example, if G has discrete series representations, do they appear in $K_G(\mathcal{B})$?*

The second problem is to study the push-forward action of the Weyl group W on $\mathrm{K}_G(\mathcal{B})$. The Weyl group action on $\mathcal{B}$ plays an essential role in [7] to obtain the Weyl character formula for compact groups in the language of KK-theory. For noncompact G, however, the Weyl group action on $\mathcal{B}$ does not commute with the G action. Fortunately, one of the advantages of K-theory is that we do not need an honest commuting action: commuting up to homotopy is enough. Hence I come to the following problem:

Problem 2. *Let T be the Cartan subgroup of G, which is noncompact in general. Give an explicit formula of the W action of $K_T(\mathcal{B})$. Moreover, figure out whether we have*

$$K_T(\mathcal{B})^W \underset{?}{\xrightarrow{\sim}} K_G(\mathcal{B}). \tag{5}$$

Other structures, such as the *Demazure operators* on $\mathrm{K}_G(\mathcal{B})$ (see [7]), is in the further plan.

3 The dg-categorification of equivariant K-theory

K-theory itself is a decategorification. However there are several reasons suggests that equivariant K-theory about needs a categrification. In [7], J. Block and N. Higson construct the globalization and localization homomorphisms for K-theory

$$\Gamma\colon \mathrm{K}_G(\mathcal{B}) \rightleftarrows \mathrm{K}_G(\text{pt}) :\Lambda \tag{6}$$

where G is semisimple and $\mathcal{B}$ is the flag variety of $G_{\mathbb{C}}$. A lot of important information about the representations of G can be found by studying Γ and Λ. For example, it is proved in [7] that

$$\begin{aligned} \Gamma \circ \Lambda =& |W| \cdot \mathrm{Id} \in \mathrm{KK}^G(\mathrm{pt}, \mathrm{pt}) \\ \Lambda \circ \Gamma =& \sum_{x \in W} I_w \in \mathrm{KK}^G(\mathcal{B}, \mathcal{B}) \end{aligned} \tag{7}$$

and this gives the Weyl character formula for compact G.

It is expected that Γ and Λ, just as the globalization and localization functors of D-modules, is an adjoint pair so that we can apply categorical tools, for example, the Barr-Beck theorem as in [5]. However, since $\mathrm{K}_G(\mathcal{B})$ and $\mathrm{K}_G(\mathrm{pt})$ are not categories, it does not make sense to talk about adjointness.

To get a reasonable categorification we need the concept of cohesive module, which is developed in [6] by J. Block.

Definition 3.1. *Let $\mathcal{A}^\bullet$ be a curved-dg algebra. the category of cohesive modules over $\mathcal{A}^\bullet$, denoted by $\mathcal{P}_{\mathcal{A}}$, is defined to be finite generated projective modules over $\mathcal{A}^0$ with connections which are compatible with the differential on $\mathcal{A}^\bullet$.*

Definition 3.2 ([8])**.** *For X and G as above, let us consider the curved-dg algebra defined as the cross product algebra*

$$\mathcal{A}^\bullet := \mathcal{S}(G; S(\mathfrak{g}^*) \otimes \Omega^\bullet(\mathcal{B})) \tag{8}$$

where the $\mathcal{S}$ denote schwarz functions. $S^i(\mathfrak{g}^*)$ has degree $2i$ and $\Omega^k(\mathcal{B})$ has degree k.

We are studying the cohesive modules $\mathcal{P}_{\mathcal{A}}$ for the above $\mathcal{A}^\bullet$, which we expect to be a dg-categorification of $\mathrm{K}_G(\mathcal{B})$. One problem we are working on is:

Problem 3. *Find a way to localize the category $\mathcal{P}_{\mathcal{A}}$ at the fixed points of the G action. This is a kind of categorification of the localization of equivariant K-theory in [20].*

4 Mackey's analogue and deformation of family algebras

The representation theory of semisimple Lie group G has another interesting constituent. Let

$$G = K \exp \mathfrak{p} \tag{9}$$

be the Cartan decomposition of G and

$$G_c = K \rtimes \mathfrak{p} \tag{10}$$

be the Cartan motion group associated to G. The Mackey's analogue is to find an identification (in various senses) of the representations of G and those of G_c.

In [13] N. Higson introduced the *spherical Hecke algebras* $\mathcal{R}(\mathfrak{g}, \tau)$ and $\mathcal{R}(\mathfrak{g}_c, \tau)$ respectively, where τ is a irreducible representation of K. These algebras have the importance that the irreducible $\mathcal{R}(\mathfrak{g}, \tau)$ modules are 1-1 correspondent to irreducible $(\mathfrak{g}, K)$-modules of G with nonzero τ-isotypical component, and the similar result holds for $\mathcal{R}(\mathfrak{g}_c, \tau)$, see [13].

For the structures of the spherical Hecke algebras, we have the following proposition:

Proposition 4.1 ([13] Propostion 2.13)**.** *For complex semisimple Lie group G, we have the following isomorphisms as algebras:*

$$\begin{aligned} \mathcal{R}(\mathfrak{g}, \tau) &\cong [U(\mathfrak{g}) \otimes \mathit{End}(V_\tau)^{op}]^K \\ \mathcal{R}(\mathfrak{g}_c, \tau) &\cong [S(\mathfrak{g}) \otimes \mathit{End}(V_\tau)^{op}]^K. \end{aligned} \tag{11}$$

The right hand sides are the *quantum family algebra* $\mathcal{Q}_\tau(\mathfrak{g})$ and the *classical family algebra* $\mathcal{C}_\tau(\mathfrak{g})$, introduced by A. A. Kirillov in [15].

In [13] Higson constructed the *generalized Harish-Chandra homomorphisms*:

$$\begin{aligned} \mathrm{GHC}_\tau &: \mathcal{R}(\mathfrak{g}, \tau) \to U(\mathfrak{h}) \\ \mathrm{GHC}_{\tau,c} &: \mathcal{R}(\mathfrak{g}_c, \tau) \to S(\mathfrak{h}) \end{aligned} \tag{12}$$

and relates them to the admissible duals of G and G_c with minimal K-type τ.

The Mackey's analogue for admissible dual has the following form:

Theorem 4.2 ([13], Section 8)**.** *Under the identification $U(\mathfrak{h}) \cong S(\mathfrak{h})$, the two homomorphisms GHC$_\tau$ and GHC$_{\tau,c}$ has the same image.*

In the end of [13], Higson proposed the problem of constructing a quantization map Q between $\mathcal{C}_\tau(\mathfrak{g})$ and $\mathcal{Q}_\tau(\mathfrak{g})$ such that the following diagram commutes.

$$\begin{array}{ccc} \mathcal{C}_\tau(\mathfrak{g}) & \overset{Q}{\dashrightarrow} & \mathcal{Q}_\tau(\mathfrak{g}) \\ \big\downarrow{\scriptstyle \mathrm{GHC}_{\tau,c}} & & \big\downarrow{\scriptstyle \mathrm{GHC}_{\tau}} \\ S(\mathfrak{h}) & \xrightarrow{\cong} & U(\mathfrak{h}) \end{array} \tag{13}$$

Here Q is a vector space isomorphism but need not to be an algebraic homomorphism.

To solve the above problem, in [23], I studied family algebras in the framework of deformation theory. First of all, $U(\mathfrak{g})$ is isomorphic to $S(\mathfrak{g})$ with the star multiplication

$$a * b = a \cdot b + \frac{1}{2}\{a, b\} + \dots.$$

For family algebras we have the same identification, first we define the Poisson bracket on $\mathcal{C}_\tau(\mathfrak{g})$:

Definition 4.1 (The Poisson bracket on $\mathcal{C}_\tau(\mathfrak{g})$)**.** *Let $\mathcal{A} = A_i \otimes a^i,\ \mathcal{B} = B_j \otimes b^j \in \mathcal{C}_\tau(\mathfrak{g})$. We define*

$$\{\mathcal{A}, \mathcal{B}\} := A_i B_j \otimes \{a^i, b^j\}. \tag{14}$$

Then it is clear that $\mathcal{Q}_\tau(\mathfrak{g})$ is isomorphic to $\mathcal{C}_\tau(\mathfrak{g})$ with the star multiplication

$$\mathcal{A} * \mathcal{B} = \mathcal{A} \cdot \mathcal{B} + \frac{1}{2}\{\mathcal{A}, \mathcal{B}\} + \dots. \tag{15}$$

It is well-known that the poisson bracket is identically zero on $S(\mathfrak{g})^K$. The generalization of this fact to family algebras is interesting. In [23] I got the following:

Theorem 4.3. *The Poisson bracket is a Hochschild 2-coboundary. In fact we can construct a Hochschild 1-cochain ∇ such that*

$$\{\cdot\,,\cdot\} = d_{Hoch}\nabla(\cdot\,,\cdot).$$

The above theorem implies that although the first order deformation from $\mathcal{C}_\tau(\mathfrak{g})$ to $\mathcal{Q}_\tau(\mathfrak{g})$ is not zero, it is infinitesimally trivial. This suggests that we can find more relations between $\mathcal{C}_\tau(\mathfrak{g})$ and $\mathcal{Q}_\tau(\mathfrak{g})$. One of the tools is the covariant Weil algebra below.

5 Covariant Weil algebras

The above problem is a natural generalization of the famous Duflo's isomorphism theorem, which gives an algebraic isomorphism $Z(U(\mathfrak{g})) \cong S(\mathfrak{g})^K$. In [1] and [2] A. Alekseev, and E. Meinrenken give a proof of Duflo's isomorphism theorem for quadratic Lie algebras, using the quantization map between the commutative Weil algebra $W(\mathfrak{g}^*)$ and the noncommutative Weil algebra $\mathcal{W}(\mathfrak{g})$.

In [22], I constructed a generalization of $W(\mathfrak{g}^*)$ and $\mathcal{W}(\mathfrak{g})$:

Definition 5.1 (covariant Weil algebras)**.**

$$\begin{aligned} W_\tau(\mathfrak{g}^*) &:= S\mathfrak{g}^* \otimes \wedge\mathfrak{g}^* \otimes \mathit{End}V_\tau \\ \mathcal{W}_\tau(\mathfrak{g}) &:= U(\mathfrak{g}) \otimes \mathit{Cl}(\mathfrak{g}) \otimes \mathit{End}V_\tau. \end{aligned} \tag{16}$$

For quadratic Lie algebra we can identify $\mathfrak{g}$ and $\mathfrak{g}^*$ hence

$$W_\tau(\mathfrak{g}^*) \cong W_\tau(\mathfrak{g}) := S\mathfrak{g} \otimes \wedge\mathfrak{g} \otimes \mathrm{End}V_\tau. \tag{17}$$

I also defined Lie derivations, contractions and differentials on $W_\tau(\mathfrak{g}^*)$ and $\mathcal{W}_\tau(\mathfrak{g})$ to make them to be curved-dg algebra with $\mathfrak{g}$ actions. In particular I have found the following form of the curvatures of $W_\tau(\mathfrak{g}^*)$ and $\mathcal{W}_\tau(\mathfrak{g})$ [22]:

Theorem 5.1. *Let e_a be a basis of $\mathfrak{g}$ and e^abe a dual basis on $\mathfrak{g}^*$. As in [1], we denote the corresponding element in $S\mathfrak{g}^*$ by v^a and denote $\tau(e_a) \in \mathit{End}(V_\tau)$ by τ_a. Then define*

$$C := v^a \tau_a \in W_\tau(\mathfrak{g}^*). \tag{18}$$

Let $d^{W,\tau}$ denote the differential on $W_\tau(\mathfrak{g}^*)$. We have

$$d^{W,\tau} \circ d^{W,\tau}(-) = [C, -] \text{ on } W_\tau(\mathfrak{g}^*). \tag{19}$$

Similarly let u_a denote the $e_a$in $U(\mathfrak{g})$ and define

$$\mathcal{C} := \frac{1}{2}(u_a u_a + 2u_a\tau_a + \tau_a\tau_a - \frac{1}{48} f_{abc} f_{abc}) \in \mathcal{W}_\tau(\mathfrak{g}). \tag{20}$$

We have

$$d^{\mathcal{W},\tau} d^{\mathcal{W},\tau}(-) = [\mathcal{C}, -] \text{ on } \mathcal{W}_\tau(\mathfrak{g}). \tag{21}$$

Since the covariant Weil algebras are generalizations of the Weil algebras, it is expected that we can construct a quantization map between $W_\tau(\mathfrak{g})$ and $\mathcal{W}_\tau(\mathfrak{g})$ which solves Higson's problem. Moreover, $\mathcal{W}_\tau(\mathfrak{g})$ have their own interests to study. One of the problems I am working on is:

Problem 4. *Find a suitable cohomology theory on covariant Weil algebras, which detects the kernel of the generalized Harish-Chandra homomorphisms and further find the quantization map of the family algebras.*

Curved-dg algebras are of increasingly importance and I expect that the covariant Weil algebra will serve as a "test" for the theory.

6 Formality and the Kashiwara-Vergne problem

The Kashiwara-Vergne conjecture is closed related to (in fact, implies) Duflo's isomorphism theorem. There are several versions of this conjecture and one of them is given as follows (see [3]):

Conjecture 6.1 (Kashiwara-Vergne conjecture)**.** *Let m_t be the star product on $S(\mathfrak{g})$ coming from Kontsevich's formality map. Let O be an open neighborhood of the origin of $\mathfrak{g}$. The Kashiwara-Vergne conjecture asserts the existence of an analytic map $\beta = \sum_i \beta^i e_i : O^2 \to \mathfrak{g}^2$, vanishing at the origin, such that*

$$\frac{d\,m_t}{dt} = -m_t \circ \sum_i \beta_t^i L(e_i). \tag{22}$$

where $\beta_t^i = t^{-1}\beta^i(tx, ty)$.

A. Alekseev and E. Meinrenken proved the above conjecture in [3] and proved its equivalence to other more algebraic versions. In [4] A. Alekseev and C. Torossian gave a proof of the generalized Kashiwara-Vergne conjecture based on the existence of Drinfeld associators.

On the other hand, in [10] and [11] V. Dolgushev introduced stable formality maps. In more details, he constructed a 2-colored operad Tam whose algebras algebras are pairs $(\mathcal{V}, \mathcal{A})$ of cochain complexes with the following data:

- a Ger_∞-structure on $\mathcal{V}$,
- an A_∞-structure on $\mathcal{A}$
- a Ger_∞-morphism from $\mathcal{V}$ to the Hochschild cochain complex $C^\bullet(\mathcal{A})$ of $\mathcal{A}$.

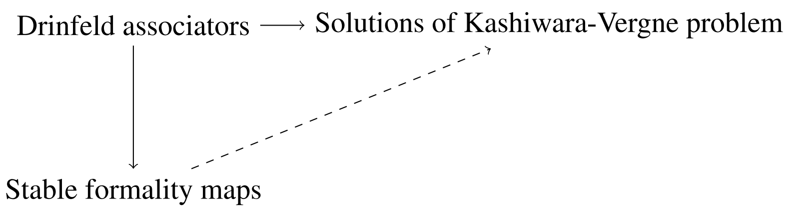
The construction of Tam also depends on a choice of Drinfeld associator.

In [10] Dolgushev defined a 2-colored operad KGra which is assembled from the graphs used by Kontsevich and acts on functions and polyvector fields. Then in [11] he construct an operad map:

$$\mathcal{U} : \mathrm{Tam} \to \mathrm{KGra} \tag{23}$$

which we called the stable formality map. Using U, for each affine space, we can construct a Ger_∞-quasi-isomorphism from $\mathcal{V}_{\mathcal{A}}$, the polyvector fields on the affine space, to the Hochschild cochains $C^\bullet(\mathcal{A})$ of the algebra of functions $\mathcal{A}$ on this affine space. The name "stable" is justified by the fact that the map $\mathcal{U}$ is valid for all (finite) dimensions.

Then we have the following commuting diagram



(24)

Now it is natural to ask whether we can find solutions for Kashiwara-Vergne problem from arbitrary stable formality maps. A project in progress by Dolgushev and me is to complete the dashed arrow in the above diagram to make it commute. Moreover, the Grothendieck-Teichmüller group acts on all three and we want the map to be compatible with the group actions.

There are some more motivations of this construction:

- This construction would give a simple way for identifying homotopy classes of stable formality quasi-isomorphisms. For the homotopies of stable formality quasi-isomorphisms see [10] Section 5.
- It may shed some light on Alekseev-Torossian conjecture and the Deligne-Drinfeld conjecture (see [12] Section 6 and [4] Section 4).

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