Quiz 6

Monday Nov 2
1. Find the limit, if it exists, or show that the limit does not exist:

\[
\lim_{(x,y)\to(0,0)} \frac{x^2 + \sin^2 y}{x^2 + y^2}.
\]

For this problem, trying \(x=0, y \to 0\), we get \(\lim \frac{\sin^2 y}{y^2} = 1\) (by L’Hospital’s Rule). \(y = 0, x \to 0\) also gives \(\lim = 1\).
To show that the limit is indeed 1, we split it as

\[
\lim_{(x,y)\to(0,0)} \frac{x^2 + \sin^2 y}{x^2 + y^2} = \lim_{(x,y)\to(0,0)} \frac{x^2 + y^2}{x^2 + y^2} - \lim_{(x,y)\to(0,0)} \frac{y^2 - \sin^2 y}{x^2 + y^2},
\]

and argue that the second part is 0 by squeeze theorem: since (check by yourself)

\[
0 \leq \frac{y^2 - \sin^2 y}{x^2 + y^2} \leq \frac{y^2 - \sin^2 y}{y^2},
\]

and

\[
\lim_{(x,y)\to(0,0)} \frac{y^2 - \sin^2 y}{y^2} = \lim_{(x,y)\to(0,0)} \frac{y^2}{y^2} - \lim_{(x,y)\to(0,0)} \frac{\sin^2 y}{y^2} = 1 - 1 = 0.
\]

2. Find the first partial derivatives of the function:

\[
f(x, y) = x^{(2y)} + 7.
\]

\[
f_x(x, y) = 2y \cdot x^{2y-1}, \quad f_y(x, y) = x^{2y} \ln x \cdot 2 = 2x^{2y} \ln x
\]

Wednesday Nov 4
1. Find the limit, if it exists, or show that the limit does not exist:

\[
\lim_{(x,y)\to(0,0)} \frac{3x^2 + \sin^2 y}{2x^2 + y^2}.
\]

When \(y=0, x \to 0\), \(\frac{3x^2 + \sin^2 y}{2x^2 + y^2} = \frac{3x^2}{2x^2} \to \frac{3}{2}\). When \(x=0, y \to 0\), \(\frac{3x^2 + \sin^2 y}{y^2} \to 1\), by L’Hospital’s Rule. So the limit does not exist.
2. Find the first partial derivatives of the function:

\[
f(x, y) = (3x)^y + 7.
\]

\[
f_x(x, y) = 3y(3x)^{y-1}, \quad f_y(x, y) = (3x)^y \ln 3x.
\]
HW8:

I graded 15.3 #64, 15.5 #32 for correctness and others for completion.

15.3
56. \(v_x x = e^{2y+xe^y}, v_x y = v_y x = e^{y+xe^y}(1 + xe^y), v_y y = e^{y+xe^y}(x + x^2e^y).\)
64. \(f_{rst} = -\frac{2}{x^2}, f_{rst} = 0.\)
70. (a) Fix \(y\) and let \(x\) vary, the level curve indicates that the value of \(f\) decreases as we move through \(P\) in the positive \(x\)-direction, so \(f_x\) is negative at \(P\).
(b) If we fix \(x\) and let \(y\) vary, the level curve indicates that the value of \(f\) increases as we move through \(P\) in the positive \(x\)-direction, so \(f_x\) is positive at \(P\).
(c) \(f_{xx} = \frac{\partial}{\partial x}(f_x), \) so \(f_{xx}\) is the rate of change of \(f_x\) as \(x\) increases. \(f_x\) is smaller ("more negative") on the left of \(P\), so \(f_x\) increases as we move through \(P\) in the positive \(x\)-direction, so \(f_{xx}\) is positive at \(P\).
(d) \(f_{xy} = \frac{\partial}{\partial y}(f_x), \) so \(f_{xy}\) is the rate of change of \(f_x\) as \(y\) increases. \(f_x\) is larger ("less negative") at points below \(P\), so \(f_x\) decreases as we move through \(P\) in the positive \(x\)-direction, so \(f_{xx}\) is negative at \(P\).
(e) \(f_{yy} = \frac{\partial}{\partial y}(f_y), \) so \(f_{yy}\) is the rate of change of \(f_y\) as \(y\) increases. \(f_y\) is larger at points above \(P\), so \(f_y\) increases as we move through \(P\) in the positive \(x\)-direction, so \(f_{xy}\) is positive at \(P\).
Slogan: 'Wider' \(\Rightarrow |f_x|\) or \(|f_y|\) smaller.

76. For each \(i, i=1,2,...,n, \) \(\frac{\partial u}{\partial x_i} = a_i e^{a_1 x_1 + ... + a_n x_n} = a_i u, \) \(\frac{\partial^2 u}{\partial x_i^2} = \frac{\partial (a_i u)}{\partial x_i} = a_i^2 u.\)
So \(\frac{\partial^2 u}{\partial x_1^2} + ... + \frac{\partial^2 u}{\partial x_n^2} = (a_1^2 + ... + a_n^2)u = u.\)
80. \(\frac{\partial T}{\partial x} = -\frac{20}{3}, \frac{\partial T}{\partial y} = -\frac{10}{3}.\)

15.4
4. \(z = 4x - 4\)
14. Show that \(f_x, f_y\) are continuous at \((3,0)\). So \(f\) is differentiable by Theorem 8. The linearization of \(f\) at \((3,0)\) is \(L(x,y) = \frac{1}{4}x + y + \frac{5}{3}.\)
32. \(dx = \Delta x = -0.04, dy = \Delta y = 0.05, z = x^2 - xy + 3y^2, z_x = 2x - y,\)
\(z_y = 6y - x.\) Thus when \(x=3, y=-1, dz = (7)(-0.04) + (-9)(0.05) = -0.73\)
while \(\Delta z = (2.96)^2 - (2.96)(-0.95) + 3(-0.95)^2 - (9 + 3 + 3) = -0.7189.\)
34. The maximal error is about \(dS = (220)(0.2) + (260)(0.2) + (280)(0.2) = 152cm^2\)
46. (a) \(f_x(0,0) = f_y(0,0) = 0.\) Approaching \((0,0)\) along the \(x\)-axis and the line \(y = x\) respectively, one can see that \(\text{lim}_{(x,y)\to(0,0)}\) does not exist, so \(f\) is discontinuous hence not differentiable there.
(b) When \((x,y) \neq (0,0), f_x(x,y) = \frac{y(y^2-x^2)}{(x^2+y^2)^2}, f_y(x,y) = \frac{x(y^2-x^2)}{(x^2+y^2)^2},\) and one can show that their limits do not exist at \((0,0).\)

15.5
10. \(\frac{\partial z}{\partial s} = e^{x+2y}\left(\frac{1}{t} - \frac{2t}{s^2}\right),\)
\(\frac{\partial z}{\partial t} = e^{x+2y}\left(\frac{2}{s} - \frac{s}{t^2}\right).\)
14. \( W - s(1, 0) = 52, W_t(1, 0) = 34. \)

22. \( \frac{\partial u}{\partial x} = \left( r \cos t + s \right) / \sqrt{r^2 + s^2} = 4/\sqrt{10}, \)
    \( \frac{\partial u}{\partial y} = \left( r + s \sin t \right) / \sqrt{r^2 + s^2} = 3/\sqrt{10}, \)
    \( \frac{\partial u}{\partial z} = \left( -r \sin t + s \cos t \right) / \sqrt{r^2 + s^2} = 2/\sqrt{10}. \)

32. \( \frac{\partial z}{\partial x} = -\frac{yz + \sin(x + y + z)}{xy + \sin(x + y + z)}, \)
    \( \frac{\partial z}{\partial y} = -\frac{xz + \sin(x + y + z)}{xy + \sin(x + y + z)}. \)

36. (a) Since \( \frac{\partial W}{\partial T} \) is negative, a rise in average temperature (while annual rainfall remains constant) causes a decrease in wheat production at the current production levels.
   (b) Since the average temperature is rising at a rate of 0.15\(^\circ\)C/year, \( dT/dt = 0.15 \), and similarly \( dR/dt = -0.1 \). Then by the Chain Rule,
   \( \frac{dW}{dt} = \frac{\partial W}{\partial T} \frac{dT}{dt} + \frac{\partial W}{\partial R} \frac{dR}{dt} = -1.1. \) Thus we estimate that wheat production will decrease at a rate of 1.1 units/year.