Rob Bayer - Practice Final

Solutions by Zvi Rosen

1. a) $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-2)^{n+1}(x+1)^{n+1}}{\sqrt[3]{(n+1)+1}} \cdot \frac{\sqrt[3]{n+1}}{(-2)^n (x+1)^n}\right|$

$$= \left|\sqrt[3]{\frac{n+1}{n+2}}\right| \cdot 2|x+1|$$

$\Rightarrow \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 2|x+1|$. For $\left|\frac{a_{n+1}}{a_n}\right| < 1$, we take $2|x+1| < 1 \Rightarrow |x+1| < \frac{1}{2} \Rightarrow \boxed{R = \frac{1}{2}.}$

Test the endpoints:

At $x = \frac{-1}{2}$, our series is $\sum_{n=1}^{\infty} \frac{(-2)^n}{\sqrt[3]{n+1}} \cdot \frac{1}{2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n+1}}$.

This series converges by the Alt. Series Test.

At $x = -\frac{3}{2}$, the series is $\sum_{n=1}^{\infty} \frac{(-2)^n}{\sqrt[3]{n+1}} \cdot \left(-\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n+1}}$.

This diverges by comparison to a p-series.

So, $\boxed{I = \left(-\frac{3}{2}, \frac{-1}{2}\right].}$

c) $\sum_{n=4}^{\infty} (-1)^n \frac{(2x+3)^n}{n^2 5^n}$. $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(2x+3)^{n+1}}{(n+1)^2 5^{n+1}} \cdot \frac{n^2 5^n}{(2x+3)^n}\right|$

$$= \left|\left(\frac{n}{n+1}\right)^2 \cdot 5 \cdot (2x+3)\right| = 5\left|\frac{n}{n+1}\right|^2 |2x+3|$$

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 5|2x+3|$. So, we need $5|2x+3| < 1$,

$\Rightarrow |x + \frac{3}{2}| < \frac{1}{10}$. $\boxed{R = \frac{1}{10}}$

Testing the endpoints,

At $x = -\frac{14}{10}$, we have $\sum_{n=4}^{\infty} (-1)^n \frac{\left(\frac{1}{5}\right)^n}{n^2 5^n} = \sum_{n=4}^{\infty} \frac{(-1)^n}{n^2}$, which converges by AST.

At $x = -\frac{16}{10}$, we have $\sum_{n=4}^{\infty} (-1)^n \frac{\left(-\frac{1}{5}\right)^n}{n^2 5^n} = \sum_{n=4}^{\infty} \frac{1}{n^2}$, which also converges as a p-series.

So, $\boxed{I = \left[-\frac{16}{10}, -\frac{14}{10}\right]}$.

3. e) $\sum_{n=1}^{\infty} \ln\left(1 + \sin\left(\frac{1}{n}\right)\right)$. We use Taylor Series.

$\sin\left(\frac{1}{n}\right) = \frac{1}{n} - \frac{1}{n^3 \cdot 3!} + \frac{1}{n^5 \cdot 5!} \cdots \Rightarrow \sin\left(\frac{1}{n}\right) \approx \frac{1}{n}$ for large n.

$\ln\left(1 + \frac{1}{n}\right) = \frac{1}{n} - \frac{1}{2 \cdot n^2} + \frac{1}{3n^3} - \cdots \Rightarrow \ln\left(1 + \frac{1}{n}\right) \approx \frac{1}{n}$.

Try LCT with $\frac{1}{n}$:

$$\lim_{n \to \infty} \frac{\ln\left(1 + \sin\left(\frac{1}{n}\right)\right)}{\frac{1}{n}} = \lim_{n \to \infty} \frac{\frac{1}{1 + \sin\left(\frac{1}{n}\right)} \cdot \cos\left(\frac{1}{n}\right) \cancel{\left(-\frac{1}{n^2}\right)}}{\cancel{\left(-\frac{1}{n^2}\right)}}$$

$= \frac{\cos(0)}{1 + \sin(0)} = 1$. Since $\sum \frac{1}{n}$ diverges, by LCT, so does $\sum \ln\left(1 + \sin\left(\frac{1}{n}\right)\right)$.

4) $\sum_{n=1}^{\infty} \ln \frac{n+2}{n} = \sum_{n=1}^{\infty} [\ln(n+2) - \ln(n)]$

$s_4 = [\cancel{\ln 3} - \ln 1] + [\cancel{\ln 4} - \ln 2] + [\ln 5 - \cancel{\ln 3}] + [\ln 6 - \cancel{\ln 4}]$

$\rightsquigarrow s_n = \ln(n+2) + \ln(n+1) - \ln(2).$

$\lim_{n\to\infty} s_n$ diverges, so the series does not converge.

5. $\sum_{n=0}^{\infty} 3^{n^2} x^{n^2} = \sum_{n=0}^{\infty} (3x)^{n^2}$. We use the root test.

$\lim_{n\to\infty} \left[(3x)^{n^2}\right]^{\frac{1}{n}} = \lim_{n\to\infty} (3x)^n$. If $|x| < \frac{1}{3}$, then this

limit is zero, so the series converges. At $x = \pm\frac{1}{3}$,

the series is $\sum_{n=0}^{\infty} 1$ or $\sum_{n=0}^{\infty} (-1)^n$, so it does not converge.

6. b) Find Taylor series at $a = 1$.

$$\frac{x-1}{(3+x)^3} = (x-1)\left[\frac{1}{(4+(x-1))^3}\right] = \frac{(x-1)}{4^3}\left[\frac{1}{\left[1+\frac{(x-1)}{4}\right]^3}\right]$$

The fraction is a variation on $\frac{1}{1+u}$.

$$\frac{d}{du}\left(\frac{1}{1+u}\right) = \frac{-1}{(1+u)^2}, \quad \frac{d}{du}\left(\frac{-1}{(1+u)^2}\right) = \frac{2}{(1+u)^3}$$

$$\frac{d}{du}\left(\sum_{n=0}^{\infty}(-1)^n u^n\right) = \sum_{n=1}^{\infty}(-1)^n \cdot n u^{n-1}, \quad \frac{d}{du}\left[\sum_{n=1}^{\infty}(-1)^n \cdot n u^{n-1}\right] = \sum_{n=2}^{\infty}(-1)^n n(n-1) u^{n-2}.$$

$$\frac{1}{(1+u)^3} = \frac{1}{2}\sum_{n=2}^{\infty}(-1)^n n(n-1)u^{n-2} \Rightarrow \frac{1}{\left(1+\frac{(x-1)}{4}\right)^3} = \frac{1}{2}\sum_{n=2}^{\infty}(-1)^n n(n-1)\left(\frac{x-1}{4}\right)^{n-2}$$

$$\Rightarrow \frac{(x-1)}{4^3}\cdot\frac{1}{\left(1+\frac{(x-1)}{4}\right)^3} = \frac{1}{2}\sum_{n=2}^{\infty}(-1)^n n(n-1)\frac{(x-1)^{n-1}}{4^{n+1}}$$

$$= \boxed{\sum_{n=1}^{\infty}(-1)^{n+1} n(n+1)\frac{(x-1)^n}{2^{2n+3}}}$$

9. c) $\displaystyle\sum_{n=1}^{\infty}\frac{n}{2^n} = \sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^n.$ Consider $\displaystyle f(x) = \sum_{n=1}^{\infty} nx^n.$

$$f(x) = \sum_{n=1}^{\infty} nx^n = x\sum_{n=1}^{\infty} nx^{n-1} = x\frac{d}{dx}\left(\sum_{n=0}^{\infty}x^n\right) = x\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{x}{(1-x)^2}$$

$$\Rightarrow f\left(\frac{1}{2}\right) = \sum_{n=1}^{\infty}\frac{n}{2^n} = \frac{1/2}{(1-1/2)^2} = 2.$$