Series Manipulations

Here's a (partial) list of some manipulations you can perform on series and what they modify:

- **Re-indexing**
  - Changes: starting place, form of $a_n$
  - Doesn’t change: value
  - Reminders: Don’t forget about “hidden” parentheses around each $n$. When you decrease the starting place, you have to increase every $n$ in $a_n$ by the same amount.
  - Example: $\sum_{n=3}^{\infty} \frac{n x^{2n+1}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(n+3)x^{2(n+3)+1}}{(2(n+3))!} = \sum_{n=0}^{\infty} \frac{(n+3)x^{2n+7}}{(2n+6)!}$

- **Factoring in/out of the $\sum$**
  - Changes: form of $a_n$
  - Doesn’t change: starting place, value
  - Reminders: You can only factor out things that don’t involve an $n$. However, as far as the sum is concerned, $x$ is a constant and can be moved in/out.
  - Example: $\frac{1}{8} \sum_{n=0}^{\infty} n^2 x^{n+2} = x^2 \sum_{n=0}^{\infty} \frac{n^2}{2^{2n-3}} = x^2 \sum_{n=0}^{\infty} \frac{x^n}{2^{2n}} = x^2 \sum_{n=0}^{\infty} \left(\frac{x}{4}\right)^n$

- **Adding/subtracting missing terms**
  - Changes: starting place
  - Doesn’t change: form of $a_n$, value
  - Reminders: Make sure you remember all the extra terms and that $x^0 = 0! = 1$
  - Example: $\sum_{n=4}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} - \sum_{n=0}^{3} \frac{x^n}{n!} = \left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right) - \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!}\right)$

Finding Sums

Finding sums of series is somewhat of an art, but here’s a general outline of the procedure I use when I’m stuck:

1. Figure out which known series you’re aiming for. The denominator is usually a good indication, but this is definitely where the “art” comes in. If there’s a factorial in the denominator, I’d start with whatever series matches that.
2. Figure out what your “$x$” is going to be. Numbers raised to an $n$ are probably going to end up being part of the $x$ you plug in at the end. Depending on your which known Maclaurin Series you’re using, sometimes this includes the $(-1)^n$ part and sometimes it doesn’t.
3. Figure out how to get from the known series to the one you have. Here’s some tricks:
   - An extra $n$ terms in the numerator usually comes from a derivative of $x^n$, $x^{2n}$, $x^{2n+1}$, etc.
   - Extra $n$ terms in a denominator usually come from integrals
   - If your series starts in the wrong place, but you like the form of $a_n$, use the third manipulation from above to fix it
   - Don’t forget about your exponent laws: $\sum x^{3n+2} = x^2 \sum (x^3)^n$
   - One thing that’s helpful is to learn to think relatively. For example, if you see $(n+1)x^n$, you should think of this as “a power of $x$ multiplied by the number one bigger than the exponent,” which is $(x^{n+1})'$
4. Start with the known series from the Maclaurin Series sheet and its function equivalent and perform the manipulations from step 2 to both sides. A few more tricks:
   - If you have an $x^{n-1}$ but want to make an $n$ show up, you can multiply both sides by $x$ and then take the derivative. Similar tricks work for denominators from integrals.
   - Make sure you do things in the appropriate order to both sides! Multiplying by $x$ and then differentiating is very different then differentiating and then multiplying by $x$.
   - If you ever integrate, remember to include a $+C$ and then solve for it.
5. Plug in the appropriate value of $x$ into both sides.