(1) **Common Initials.**

(a) How many people need to be members of a group before we can be certain that two people have the same first and last initials in English?

Product rule gives us the number of pigeonholes, which is 26^2, and the pigeonhole principle gives us $26^2 + 1$ people.

(b) What if we require that they have the same first, middle and last initials? Some people may have no middle name, in which case we count "Blank" as a permitted middle initial.

Product rule again gives us the number of pigeonholes, which is $26^2 \times 27$, and the pigeonhole principle gives us $26^2 \times 27 + 1$ people.

(2) **Types of Fruit.** Suppose a bag contains unlimited numbers of 1) apples, 2) bananas, 3) oranges, and 4) strawberries.

(a) How many fruit must you draw at random from the bag before you know that you have 4 fruit of the same type?

Let the 4 types of fruit be pigeonholes, and the N selections be pigeons. The Generalized Pigeonhole Principle says that we can only guarantee $\lceil N/4 \rceil$ in a given pigeonhole. So for $\lceil N/4 \rceil \geq 4$, we need $N \geq 13$.

(b) Suppose you drew 5 apples, 2 bananas, 4 oranges, and 2 strawberries. In how many different orders could you have picked those fruit?

Use the formula for permutations with repeated letters. There are $13!/(5! \cdot 2! \cdot 4! \cdot 2!)$.

(c) Suppose I want 4 of the same type OR 4 of all different types (i.e. at least one of each type) How many fruit do I need to pick?

Since having one of each type achieves the goal, we assume that only 3 pigeonholes are used. By the generalized Pigeonhole Principle we need $N \geq 10$ to have $\lceil N/3 \rceil \geq 4$. So the answer is 10.

(3) **Euler's phi function.** Recall that $\varphi(n) =$ the number of integers m such that $1 \leq m \leq n$ and the greatest common divisor of m and n is 1. The strategy here is to use inclusion and exclusion. If $p_1, \ldots, p_k$ are the prime factors of n, then $\varphi(n) = |(A_{p_1} \cap \cdots \cap A_{p_k})^C|$, where A_p is defined as the set of multiples of p less than or equal to n.

(a) $\varphi(120)$ $|(A_2 \cap A_3 \cap A_5)^C| = 120 - \frac{120}{2} - \frac{120}{3} - \frac{120}{5} + \frac{120}{6} + \frac{120}{10} + \frac{120}{15} - \frac{120}{30} = 32$.

(b) $\varphi(p)$ for p prime. Every number $k < p$ has $\gcd(k, p) = 1$ so $\varphi(p) = p - 1$.

(c) $\varphi(2^n)$ for n an integer. $p = 2$ is the only prime factor, so we have $\varphi(2^n) = 2^n - \frac{2^n}{2} = 2^{n-1}$.

(d) $\varphi(10^n)$ for n an integer. $|(A_2 \cap A_5)^C| = 10^n - \frac{10^n}{2} - \frac{10^n}{5} + \frac{10^n}{10} = 10^{n-1}(10 - 2 - 5 + 1) = 4 \cdot 10^n$.