1. Probability

(1) What is the probability of drawing a blackjack (A and one of 10, J, Q, or K) when you pick two cards from a standard deck?

We define our probability space Ω as the set of unordered pairs of cards from a standard deck.

Our hand is determined by the suit of the Ace, and the other card from the eligible set. By the product rule,

$$|A| = 4 \cdot 16 = 64.$$

As for the probability space,

$$|\Omega| = \binom{52}{2} = \frac{52 \cdot 51}{2}.$$

Therefore,

$$P(A) = 64/\binom{52}{2} = \frac{64 \cdot 2}{51 \cdot 52}.$$

(2) What is the probability of getting a straight when you pick a hand of five cards? (In this context, a straight counts with Ace low or Ace high, but not as the middle of a straight)

We assume that straight flushes are straights for this question.

We define our probability space Ω as the set of unordered sets of 5 cards from a standard deck.

We denote by S the set of straights. To count the straights, we count the possible values that the five cards may take: $\{\{A, 2, 3, 4, 5\}, \ldots, \{10, J, Q, K, A\}\}$ – this gives 10 possible sets. As for the suits, no restrictions are placed on this, so by the product rule (noting that each card has a distinct value), we have 4^5 possible choices of suits.

Therefore, $|S| = 10 \cdot 4^5$. Since $|\Omega| = \binom{52}{5}$, we have $P(S) = 10 \cdot 4^5/\binom{52}{5}$.

2. Conditional Probability

(3) Rachel rolls two standard dice. What is the probability that she rolled at least one 6, given that she rolled two distinct numbers?

We define our probability space Ω as the set of ordered pairs of die rolls, which is an equal probability space.

Let A denote the set of roll pairs with at least one six, and B denote the set of pairs of distinct rolls. $A \cap B$ can be described as the set of roll pairs with *exactly* one six. We are interested in $P(A|B)$ which can be found using the formula:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

By the product rule, $|\Omega| = 6 \cdot 6 = 36$.

We have two choices in defining an element of $A \cap B$: which slot to put the six in, and what the other roll should be. By the product rule, this means $|A \cap B| = 2 \cdot 5 = 10$.

As for B, these are determined by ordered pairs of distinct elements of $\{1, \ldots, 6\}$, i.e. $P(6, 2) = 30$.

Using the fact that we are in an equal probability space, this defines the probabilities in question, so:

$$P(A|B) = \frac{10/36}{30/36} = 1/3.$$

(4) A string of 10 letters in the English alphabet is chosen uniformly at random. What is the probability that the string includes an A given that it includes exactly 3 vowels? ("y" is not considered a vowel.)

Let $\Omega =$ the set of 10-letter strings; the selection is uniformly random, so this is an equal probability space. Note that $|\Omega| = 26^{10}$.

Let A denote the set of strings with an "a", and B denote the set of strings with exactly 3 vowels. We want $P(A|B) = P(A \cap B)/P(B)$.

First we count the strings in B. $|B| = \binom{10}{3} \cdot 5^3 \cdot 21^7$, choosing first where to put the vowels, next what the vowels are, and finally what the consonants are (by the product rule).

$A \cap B =$ strings with exactly 3 vowels containing an "a". Let C denote $B \setminus (A \cap B)$ (i.e. the complement of $A \cap B$ in B), the set of strings in B with no "a"s. By the same reasoning as above, $|C| = \binom{10}{3} \cdot 4^3 \cdot 21^7$, changing the 5 to a 4 since we now cannot use "a". So,

$$|A \cap B| = \binom{10}{3} \cdot 21^7 \cdot (5^3 - 4^3).$$

$$\Rightarrow P(A|B) = \frac{\binom{10}{3} \cdot 21^7 \cdot (5^3 - 4^3)/26^{10}}{\binom{10}{3} \cdot 21^7 \cdot 5^3/26^{10}.}$$

$$\Rightarrow P(A|B) = \frac{5^3 - 4^3}{5^3}.$$

Note that the consonants end up being essentially irrelevant here, since they don't affect the appearance of an "a".

3. Bayes' Theorem

(5) In a given population, 10% of families have 1 child, 25% have two children, 35% have three children, and 30% have four children.

A family is selected at random from the population, and a child is chosen at random from that family. If the child chosen is the oldest child, calculate the probability that the chosen family has k children, for $k = 1$ and 4.

First we define the probability space and the subsets of interest. Let Ω be the set of pairs of selected family and selected child from that family e.g. an event might be (Johnsons, Sarah).

The subsets will be given by: B_1 = set of pairs where chosen family has 1 child, B_2 = set of pairs where chosen family has 2 children, B_3 = set of pairs where chosen family has 3 children, and B_4 = set of pairs where chosen family has 4 children. Finally, A = set of pairs where the chosen child is the oldest in the family.

We want to obtain $P(B_k|A)$; to get there, we will apply the version of Bayes' Theorem for multiple cases. First, we make a list of known probabilities:

$P(B_1) = 0.1$	$P(A|B_1) = 1$
$P(B_2) = 0.25$	$P(A|B_2) = 1/2$
$P(B_3) = 0.35$	$P(A|B_3) = 1/3$
$P(B_4) = 0.3$	$P(A|B_4) = 1/4$

The right-hand column is obtained by noting that the choice of children is random from an equal probability space of size $1, 2, 3$ or 4. To get our answer, we plug into Bayes' Theorem:

$$P(B_1|A) = \frac{P(A|B_1)P(B_1)}{P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + P(A|B_3)P(B_3) + P(A|B_4)P(B_4)}$$
$$= \frac{1 \cdot 0.1}{1 \cdot 0.1 + (1/2) \cdot 0.25 + (1/3) \cdot 0.35 + (1/4) \cdot 0.3} = 0.24$$

Similarly, for $k = 4$, we have

$$P(B_4|A) = \frac{P(A|B_4)P(B_4)}{P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + P(A|B_3)P(B_3) + P(A|B_4)P(B_4)}$$
$$= \frac{(1/4) \cdot 0.3}{1 \cdot 0.1 + (1/2) \cdot 0.25 + (1/3) \cdot 0.35 + (1/4) \cdot 0.3} = 0.18$$