Notes were taken by Zvi Rosen, and images were coded by Alejandro Morales.
Lecture notes regarding the Positive Grassmannian are available at
`math.mit.edu/~apost/courses/18.318/`.

## 1. Tuesday, June 12, 2012 / Matroids

Let $v_1, \ldots, v_n \in \mathbb{R}^k, n \geq k$. Suppose we have a linear dependence

$$v_1 + 25v_2 = 271v_8.$$

In this context, the coefficients in the equation are not important, just the variables.

**Example 1.1.** $k = 2$. See Figure 1.

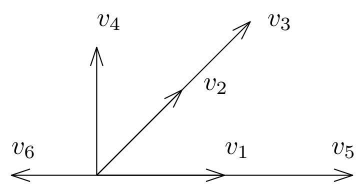



Figure 1. Rank 2 Matroid

In this example, $v_1||v_5||v_6, v_2||v_3$, and $v_4$ are parallel classes of vectors.

**Example 1.2.** $k = 3$. Consider the vectors being projected towards a plane from a point, as in Figure 2.

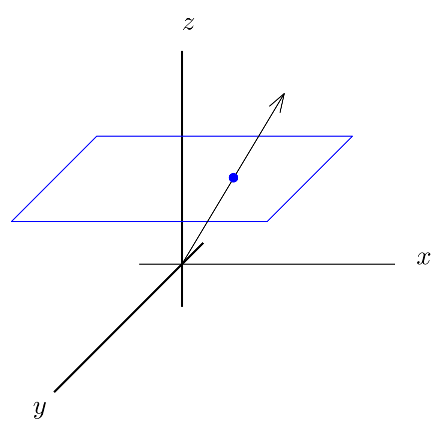



Figure 2. Vectors Intersecting the Plane

For a sample configuration, see Figure 3.

A $k \times n$ matrix has column vectors $v_1, \ldots, v_n \in \mathbb{R}^k$. Let us denote the maximal minor (i.e. determinant of submatrix) defined by these columns with $\Delta_I$ where $I = \{i_1, \ldots, i_k\}$.

Then, $v_{i_1}, \ldots, v_{i_k}$ are linearly dependent $\Leftrightarrow \Delta_I = 0$.

**Example 1.3.**

$$A = \begin{pmatrix} 1 & 1 & 3 & 1 & 1 \\ 1 & 2 & 2 & 1 & 1 \\ 1 & 3 & 1 & 0 & 0 \end{pmatrix}$$

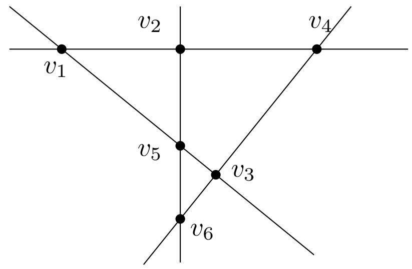



FIGURE 3. Rank 3 Matroid

Observe that $v_2 + v_3 = 4v_1$.

$$\Rightarrow \Delta_{123} = \begin{vmatrix} 1 & 1 & 3 \\ 1 & 2 & 2 \\ 1 & 3 & 1 \end{vmatrix} = 0.$$

On the other hand,

$$\Delta_{124} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 3 & 0 \end{vmatrix} \neq 0$$

implies that $v_1, v_2, v_4$ are independent.

**Notation 1.4.** Let $[n]$ denote the set $\{1, 2, \ldots, n\}$. Let $\binom{[n]}{k}$ be the set of $k$-element subsets of $[n]$.

**Definition 1.5.** Let

$$\mathcal{M}_A = \left\{ I \in \binom{[n]}{k} \middle| \Delta_I(A) \neq 0 \right\}.$$

**Example 1.6** (Relations between the $\Delta_I$'s)**.** Consider a generic matrix of size $2 \times 4$.

Let

$$A = \begin{pmatrix} a & b & c & d \\ e & f & g & h \end{pmatrix}.$$

We actually have a relation:

$$\Delta_{13}\Delta_{24} = \Delta_{12}\Delta_{34} + \Delta_{14}\Delta_{23}.$$

This is a consequence of Ptolemy's Theorem regarding quadrilaterals inscribed in circles.

In general, relations of this type are called Plücker relations.

**Notation 1.7.** Let $\sigma \in S_n$. Then,

$$\Delta_{a_1 \cdots a_n} = sgn(\sigma)\Delta_{\sigma(a_1)\cdots\sigma(a_n)}.$$

If $a_i = a_j$ for any $i \neq j$, then

$$\Delta_{a_1 \cdots a_n} = 0.$$

**Lemma 1.8** (Sylvester's Lemma)**.**

$$\Delta_{i_1 \cdots i_k} \cdot \Delta_{j_1 \cdots j_k} = \sum \Delta_{i'_1 \cdots i'_k} \cdot \Delta_{j'_1 \cdots j'_k}$$

*where $\{i'_1, \cdots, i'_k\}$ and $\{j'_1, \cdots, j'_k\}$ are obtained from the original indices by switching $i_1$ with some $j$'s.*

*More explicitly, given $I, J \in \binom{[n]}{k}$, and $i \in I$,*

$$\Delta_I \Delta_J = \sum_{j \in J} \pm \Delta_{(I \setminus i) \cup j} \Delta_{(J \setminus j) \cup i}.$$

Given these Plücker relations, we can learn about independent sets of vectors. For example, if $\{12\}, \{23\}$ are dependent, then $\Delta_{12} = \Delta_{23} = 0$. Then, since

$$\Delta_{13}\Delta_{24} = \Delta_{12}\Delta_{34} + \Delta_{14}\Delta_{23},$$

either $\Delta_{13}$ or $\Delta_{24} = 0$, implying that at least one of these pairs is also dependent.

**Definition 1.9.** A matroid of rank $k$ on $n$ elements is a non-empty subset $\mathcal{M} \subset \binom{[n]}{k}$. Elements $I \in \mathcal{M}$ are called bases of $\mathcal{M}$ if they satisfy the exchange axiom:

**Exchange Axiom**: $\forall I, J \in \mathcal{M}, i \in I$, there exists $j \in J$ such that

$$(I \setminus \{i\}) \cup \{j\} \in \mathcal{M}.$$

**Theorem 1.10.** *For $k \times n$ matrix of rank $k$, $\mathcal{M}_A$ is a matroid.*

Not every matroid is representable as a matrix.

**Example 1.11** (Fano Matroid)**.** See Figure 4. It satisfies the exchange axiom, but it is not representable.

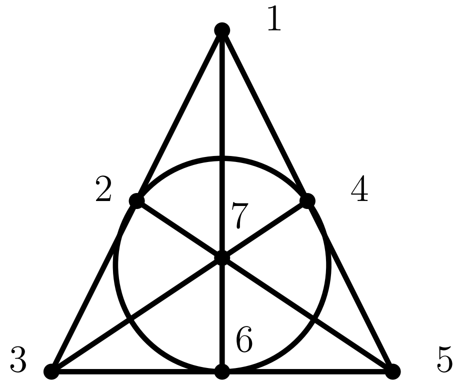



Figure 4. The Fano Matroid

**Example 1.12** (Almost Pappus Matroid)**.** Recall the Pappus theorem from geometry regarding points on a pair of lines. (see Figure 5)

Including the central line posited by the theorem, the matroid is representable. Remove the relation induced by the central line. The resulting matroid of rank 3 is not representable.

A similar thing happens when we use Desargues' Theorem. The matroid *with* the predicted line is representable, *without* it is not representable.

**Example 1.13.** Let $*$ denote a generic element. Let

$$A = \begin{pmatrix} * & * & * & * \\ * & * & 0 & 0 \end{pmatrix}.$$

Then,

$$\mathcal{M}_A = \binom{[4]}{2} \setminus \{34\}.$$

**Example 1.14.** Figure 5. We start with a few values of $k$.

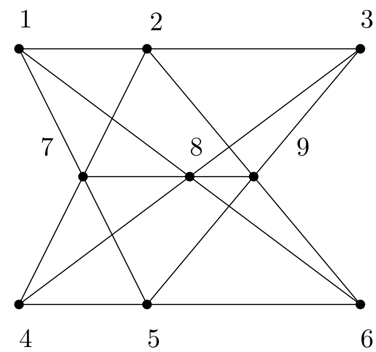



Figure 5. The Pappus Matroid

## 2. Wednesday, June 13, 2012

### Matroid Polytopes

**Definition 2.1.** A *convex set* is a set $S \subset \mathbb{R}^n$, such that for any $A, B \in S$, the line $[A, B] \subset S$.

**Definition 2.2.** A *convex polytope* is the convex hull of, i.e. the minimal convex set containing, some finite set of points.

**Definition 2.3.** Let $I = \{i_1, \ldots, i_n\} \subset [n]$. Let $v_I$ be a vector with ones in he $i_j$-th position, and zeros everywhere else. Given a matroid $\mathcal{M}$, the *matroid polytope* $P_{\mathcal{M}}$ is the polytope whose vertices are $v_I$ for all $I \in \mathcal{M}$.

**Example 2.4.** Let $\mathcal{M} = \binom{[4]}{2}$. Then $P_{\mathcal{M}}$ has six vertices: $(1,1,0,0), (1,0,1,0), \ldots$. This results in an octahedron (Figure 6).

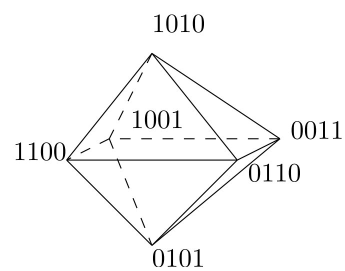



Figure 6. Matroid Polytope for $\mathcal{M}$

The edges of the octahedron are those given by a single switch of a zero and a one. This is as a result of the exchange axiom.

**Remark 2.5.** The edge $[v_I, v_J]$ is an edge of the polytope iff $J = (I \setminus i) \cup j$ for some $i, j$.

Taking only the vectors corresponding to a pyramid in the octahedron above, we obtain another matroid polytope, as in Figure 7.

However, taking only four vectors in a tetrahedron arrangement, as in Figure 8, does not yield a matroid polytope, since the edge connecting 1100 to 0011 does not involve a single exchange.

### Positroids

**Definition 2.6.** A positive matroid, or positron, is a matroid $\mathcal{M}$ represented by a $k \times n$ matrix $A$ such that $\Delta_I(A) \geq 0$ for all $I \in \mathcal{M}$.

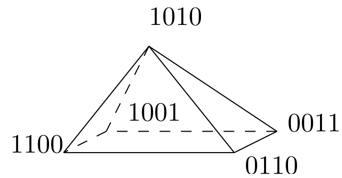



Figure 7. Pyramid as Matroid Polytope.

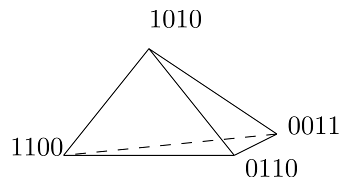



Figure 8. Tetrahedron, a non-Matroid Polytope.

**Example 2.7** (k = 2, n = 4)**.** Ptolemy's relation:

$$\Delta_{13}\Delta_{24} = \Delta_{12}\Delta_{34} + \Delta_{14}\Delta_{23}.$$

The exchange axiom is equivalent to saying that if one term is $\neq 0$, then one other term $\neq 0$. On the other hand, for positroids, if one of the terms on the right-hand side is nonzero, then the left-hand side must also be nonzero. The three diagonals of the octahedron are

$$D_1 = [v_{13}, v_{24}] \qquad \to \text{left-hand side}$$
$$D_2 = [v_{12}, v_{34}] \qquad \to \text{first right-hand term}$$
$$D_3 = [v_{14}, v_{23}] \qquad \to \text{second right-hand term.}$$

Therefore, if a matroid polytope $P$ contains one diagonal then it should contain another diagonal.

A positroid has the extra condition that if $P$ contains one of the horizontal diagonals $D_2$ or $D_3$, then it should contain the vertical diagonal $D_1$.

**Example 2.8.** Let $\mathcal{M} = \binom{[4]}{2} \setminus \{13\}$. The resulting polytope, displayed in Figure 9, is a matroid polytope but not a positroid polytope.

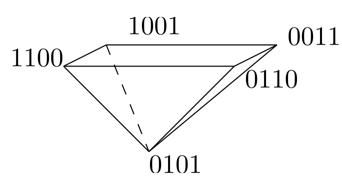



Figure 9. Non-Positroid Polytope.

**Theorem 2.9.** *Under cyclical shift of the indices, i.e. via the permutation* $(12\cdots n)$*, a positroid remains a positroid.*

*Proof.* Let $A = [v_1 v_2 \ldots v_n]$, and suppose $\Delta_I(A) \geq 0$. Then $A' = [v_2 v_3 \ldots (-1)^{n-1} v_1]$ must also have $\Delta_I(A') \geq 0$ since all of the maximal minors not involving $v_1$ have the same orientation, and any minor involving $v_1$ will be corrected by the appropriate sign. □

**Definition 2.10.** A *decorated permutation* is a permutation $\pi = \binom{12\cdots n}{\pi_1 \pi_2 \cdots \pi_n}$ with fixed points $\pi_i = i$ colored in two colors.

**Example 2.11.**

$$\pi = \binom{123456}{315{\color{red}4}2{\color{blue}6}}$$

This permutation is depicted graphically in Figure 10. Let $k =$ the number of arcs directed to the left in the diagram.

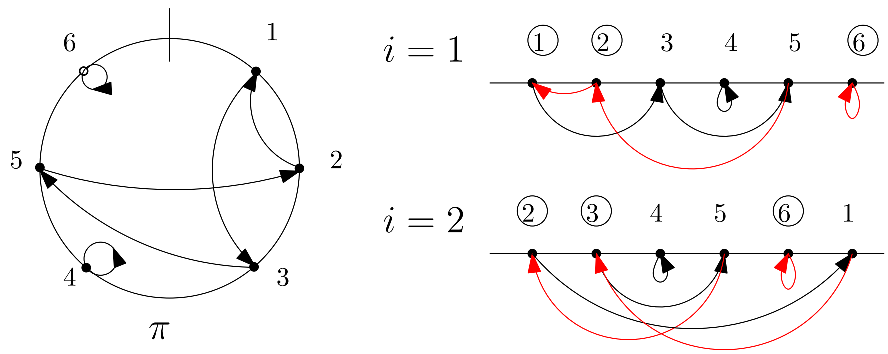



FIGURE 10. Decorated Permutation.

**Proposition 2.12.** *Positroids of rank $k$ on $n$ elements are in bijection with decorated permutations of size $n$ with $k$ left arcs.*

**Exercise 2.13.** Cut the circle between $i-1$ and $i$. Show that the number of left arcs is the same for all $i$'s.

**Definition 2.14.** Let $\pi$ be a permutation. A (weak) exceedance in $\pi$ is an index $i$ s.t. $\pi_i \geq i$.

**Example 2.15.**

$$\pi = \binom{123456}{524163}.$$

This permutation has four exceedances.

**Definition 2.16.** Let $A_{kn}$ = the number of permutations of size $n$ with $k$ exceedances $\pi_i \geq i$.

**Theorem 2.17.** *The number of positroids of rank $k$ on $n$ elements is given by:*

$$A_{k,n} + nA_{k,n-1} + \binom{n}{2} A_{k,n-2} + \cdots$$

$$= \sum_{l \geq 0} \binom{n}{k} A_{k,n-l}.$$

These numbers $A_{k,n}$ are the Eulerian numbers, i.e. the number of permutations with $k-1$ descents. A descent is an index $i$ such that $w_i > w_{i+1}$.

**Example 2.18** (Permutations on 3)**.**

| | Descents | Exceedances |
|---|---|---|
| 123 | 0 | 3 |
| 132 | 1 | 2 |
| 213 | 1 | 2 |
| 231 | 1 | 2 |
| 312 | 1 | 1 |
| 321 | 2 | 2 |

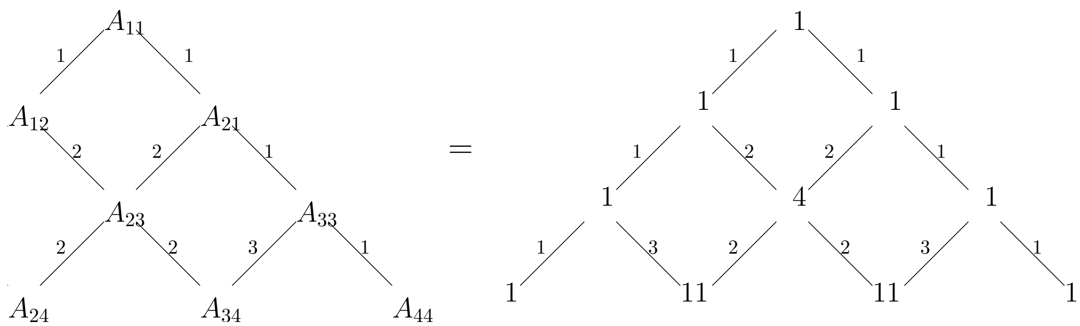



FIGURE 11. Euler Triangle.

See Figure 11 for a Pascal-type triangle of Eulerian numbers.

Label each left branch with natural numbers starting with 1 at the leftmost, then do the same for each right branch starting with 1 at the rightmost. The child of two values will be the sum of each parent multiplied by the number of the branch leading from that parent to the child.

## 3. Thursday, June 14, 2012

**Definition 3.1.** Let $N_n$ = number of decorated permutations of size $n$.

Let $D_n$ = number of derangements, i.e. permutations without fixed points, of size $n$.

**Example 3.2.** For $n = 3$, we have (231) and (312) so $D_3 = 2$.

**Theorem 3.3.**

$$D_n = n!(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots).$$
$$N_n = n!(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots).$$

**Corollary 3.4.**

$$\lim_{n\to\infty} \frac{D_n}{n!} = \frac{1}{e}.$$
$$\lim_{n\to\infty} \frac{N_n}{n!} = e.$$

**Definition 3.5.** Let $D_n(q)$ = the number of permutations in $S_n$ with $q$-colored fixed points.

$$D_n(q) = \sum_{\pi\in S_n} q^{\text{\# fixed points in } \pi}.$$

Here is the proof of the Theorem:

*Proof.* Suppose $\pi$ has fixed points colored in colors $\{1, 2, 3, \ldots, 9\}$. Erase all fixed points with colors $\{2, 3, \ldots, 9\}$. This gives us a usual permutation of size $n - r$.

$$D_n(q) = \sum_{r=0}^{n} \binom{n}{r} (q-1)^r (n-r)!.$$

$$= \sum \frac{n!}{r!(n-r)!}(q-1)^r(n-r)! = n!\left(1 + \frac{(q-1)}{1!} + \frac{(q-1)^2}{2!} + \cdots + \frac{(q-1)^n}{n!}\right).$$

□

We also have recurrence relations: $D_n = nD_{n-1} + (-1)^n, N_n = nN_{n-1} + 1$.

Returning to matroids, consider a $k \times n$ matrix $A$ with $\Delta_I(A)$ maximal minors. Row operations do not change $\mathcal{M}_A$; therefore, we can reduce it to a simple form – the row-echelon form.

**Example 3.6.**

$$\begin{pmatrix} 1 & * & 0 & 0 & * & * & 0 & * & 0 \\ 0 & 0 & 1 & 0 & * & * & 0 & * & 0 \\ 0 & 0 & 0 & 1 & * & * & 0 & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

In this example, we call columns 1,3,4,7,9 *pivot columns*.

$$\Rightarrow \Delta_{1,3,4,7,9} = 1 \neq 0.$$

$I = \{1, 3, 4, 7, 9\}$ is the lexicographically minimal base of the matroid $\mathcal{M}_A$.

Leaving out the pivot columns, we get a matrix that looks like a Young diagram:

$$\begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The vertical steps in the Young diagram represent the pivot columns of the matrix that were excluded.

Consider the positroid $\mathcal{M}$ given by the matrix

$$A = [v_1 \ \cdots \ v_n] \qquad \Delta_I(A) \geq 0.$$

**Example 3.7** (k = 2)**.** For $k = 2$, we have $\Delta_{ij} \geq 0 \Rightarrow i < j$ requires $v_j$ to be counterclockwise from $v_i$ (with the positive $x$-axis as the boundary).

Let the vectors $v_1, \ldots, v_8$ be situated in the upper half plane according to Figure 12.

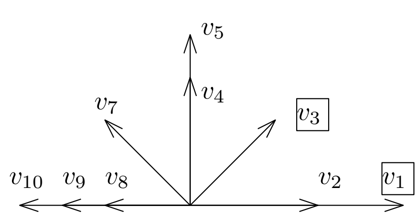



FIGURE 12. Vector Positroid.

After row reducing and deleting the pivot columns, we have a matrix that looks like:

$$\begin{pmatrix} r & q & p & t & 0 & 0 & z & x \\ 0 & 0 & 0 & s & 0 & u & y & \end{pmatrix}$$

Better, we write it as:

$$\begin{pmatrix} \bullet & \bullet & \bullet & \bullet & 0 & 0 & \bullet & \bullet \\ 0 & 0 & 0 & \bullet & 0 & \bullet & \bullet & \end{pmatrix}$$

A blocked zero is one with a dot over it. Everything to the left of a blocked zero is zero. This translates the geometric rule from above into a combinatorial rule: the pattern of having a 0 with a dot above and a dot to the left is forbidden. This is indeed the positroid condition for all values of $k$.

**Example 3.8** (k = 3)**.** Consider the affine representation of a matroid in $\mathbb{R}^3$. (See Figure 13 for an example.)

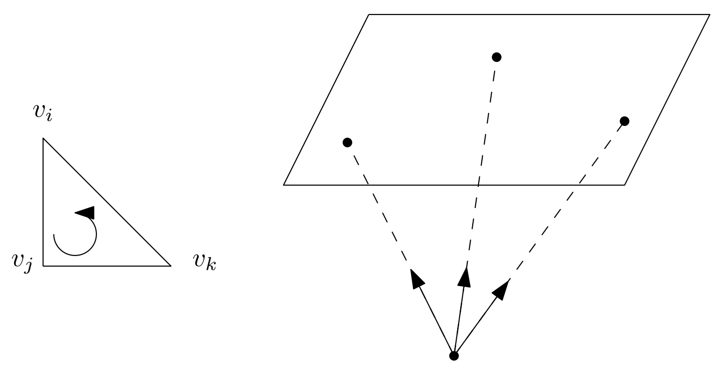



FIGURE 13. Affine Representation of a Matroid.

$\Delta_{ijk} \geq 0, i < j < k \Rightarrow$ any triangle has to have increasing index in counter-clockwise direction. This implies that the set of all vertices must be on the boundary of a convex polygon.

See Figure 14 for an example, along with the corresponding matrix shape.

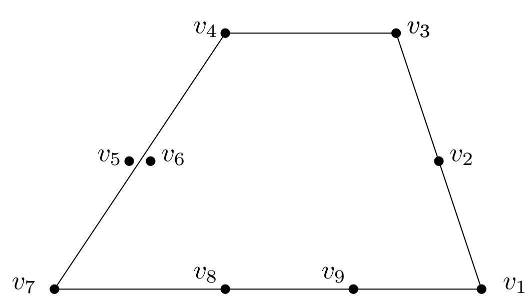



FIGURE 14. Representation of Positroid.

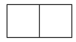
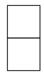
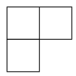
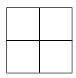
The geometric condition again translates to the blocked zero condition.

**Definition 3.9.** A hook diagram is a filling of a Young diagram by dots and zeros without the forbidden pattern described in the examples.

Another characterization: At every dot in the diagram draw a line downward and a line to the right; there should be a dot at every point of intersection.

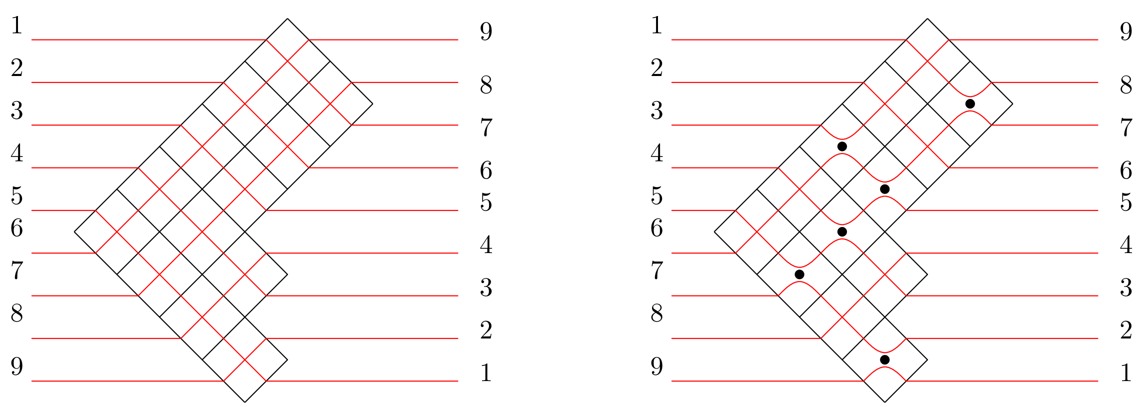
**Theorem 3.10.** *Positroids of rank $k$ on $n$ elements are in bijection with hook diagrams that fit inside a $k \times n - k$ rectangle.*

**Example 3.11** (k = 2, n = 4).

| Young Diagram | #Hook Diagrams |
|---|---|
| $\emptyset$ | 1 |
|  | 2 |
|  | 4 |
|  | 4 |
|  | 8 |
|  | $16 - 2 = 14$ |

The total number of hook diagrams is 33, which we can also obtain from examination of the matroid polytope.

See Figure 15 for a description of the bijection between hook diagrams and decorated permutations. Alternatively, a trick with mirrors is demonstrated in Figure 16.




$$\begin{array}{cccccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ & 3 & 1 & 6 & 2 & 4 & 8 & 5 & 7 & 9 \\ \pi = & 5 & 3 & 1 & \boxed{4} & 7 & 6 & 9 & 2 & 8 \end{array}$$

FIGURE 15. Pipe Dreams.

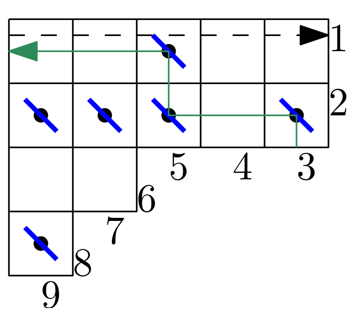



FIGURE 16. Mirrors.

## 4. Friday, June 15, 2012

**Definition 4.1.** *Plabic (planar bicolored) graphs* are trivalent graphs on vertices colored in 2 colors embedded in a disk with $n$ boundary points.

See Figure 17 for an example.

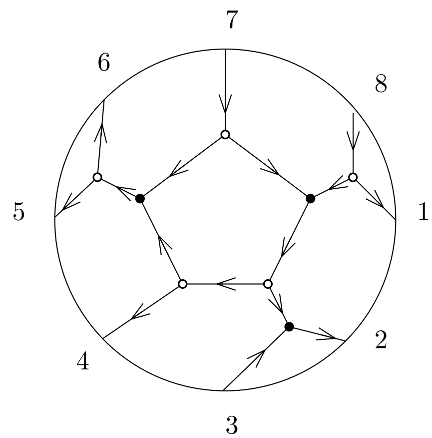



Figure 17. Plabic Graph.

**Definition 4.2.** A perfect orientation of $G$ gives each white vertex one incoming edge and two outgoing edges, and each black vertex two incoming edges and one outgoing edge. For a perfect orientation $P$, let $I_P =$ the set of boundary edges directed inside the disk.

**Theorem 4.3.** *Let $\mathcal{M}_G = \{I_P \,|\, P$ is a perfect orientation$\}$. $\mathcal{M}_G$ is a matroid.*

**Example 4.4** (n = 4)**.** This plabic graph has one white vertex and one black vertex. The resulting matroid $\mathcal{M}_G = \binom{[4]}{2} \setminus \{14\}$.

**Proposition 4.5.** *$\mathcal{M}_G$ has rank $k$ s.t.*

$$k - (n-k) = \#\ \textit{black vertices}\ - \#\ \textit{white vertices}\ .$$

**Theorem 4.6.** *The class of matroids $\mathcal{M}_G$ is exactly the class of all positroids.*

**Example 4.7.** The set of plabic graphs is not in bijection with the set of matroids. For an example of two graphs that return the same matroid, see Figure 18.

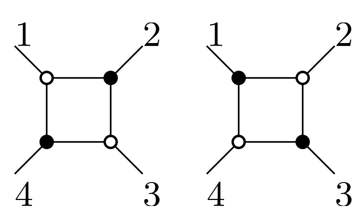



Figure 18. Graphs with the Same Matroid.

There are allowed local moves of plabic graphs that preserve the matroid: see Figure 19.

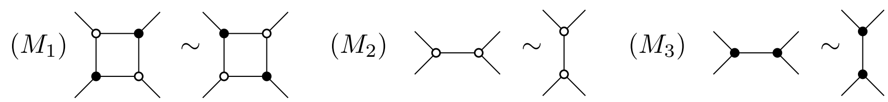



FIGURE 19. Allowed Local Moves.

**Definition 4.8.** A *Wiring diagram* is a way to represent a permutation $\pi \in S_n$ in a diagram that includes path crossings. For example, the permutation (4213) is displayed in Figure 20.

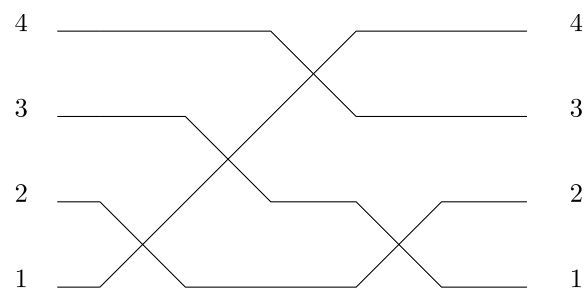



FIGURE 20. Wiring Diagram for (4213).

In a similar way to the above, there are local moves of wiring diagrams that do not affect the permutation. See Figure 21.

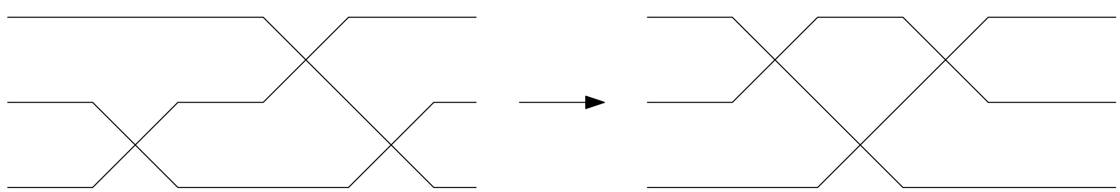


FIGURE 21. Wiring Moves.

You can map a wiring diagram to a plabic graph by sending each wire cross to a pair of vertices – on top, a white vertex, and connected to it by an edge below, a black vertex, as in Figure 22 Of course, this does not describe the entire class of plabic graphs, only a very special class of them.

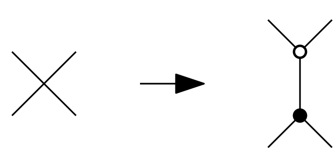


FIGURE 22. Wiring to Plabic Graph.

Figure 23 illustrates how given two wiring diagrams for the same permutation, one can use the allowed local moves of plabic graphs to get from one to the other.

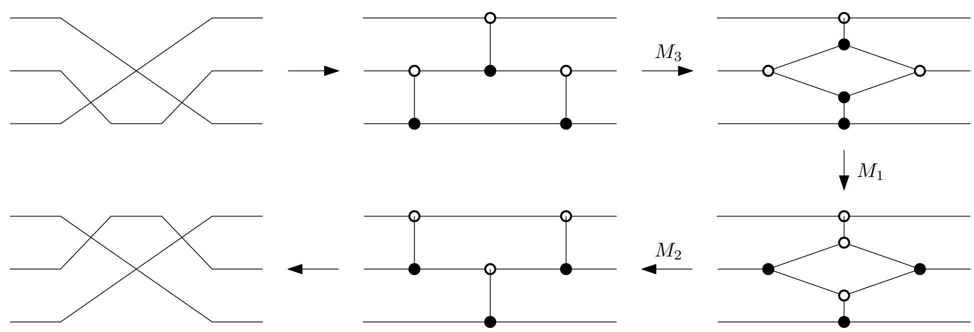



FIGURE 23. Wiring Diagrams and Local Moves.

**Definition 4.9.** Reduced plabic graphs:

(1) We do not allow loops, except for boundary lollipops.
(2) We do not have double edges.
(3) It is also impossible to create a loop or a double edge by the local moves.

(See Figure 24 for images)

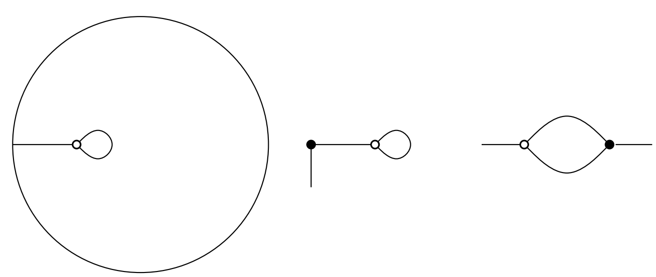


FIGURE 24. Boundary lollipop allowed. Double edge & loop not allowed.

**Theorem 4.10.** *{Positroids} $\overset{\sim}{\longleftrightarrow}$ {Reduced Plabic Graphs} / local moves.*

There exists a map from {Plabic Graphs} $\to$ {Decorated Permutations}. It reads the edges of the graph as roads, and the colors of the vertices as traffic lights: a white vertex means "turn left" and a black vertex means "turn right." See Figure 25 for a demonstration.

**Theorem 4.11.** *Local moves of $G$ don't change the corresponding decorated permutation.*

**Summary:** The following objects are in bijection (with known explicit bijections):

(1) Positroids.
(2) Decorated Permutations.
(3) Hook Diagrams.
(4) Reduced Plabic Graphs modulo the local moves.

**Definition 4.12.** A $q$-coloring of $G$ is a way to color the vertices in $q$-colors $1, 2, \ldots, q$ such that vertices connected by an edge have different colors.

**Definition 4.13.** Let $\chi_G(q)$ = the number of $q$-colorings of $G$. This is the chromatic polynomial of the graph $G$.

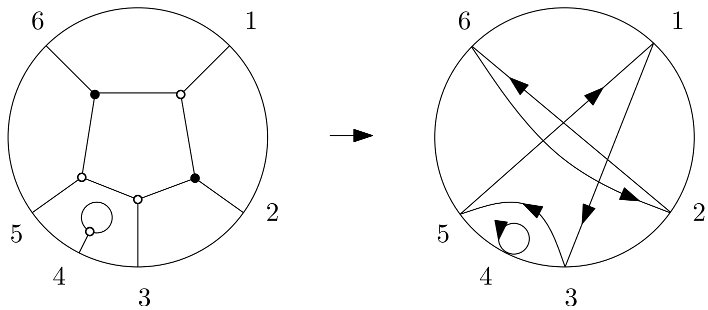



Figure 25. Plabic Graphs as Road Maps.

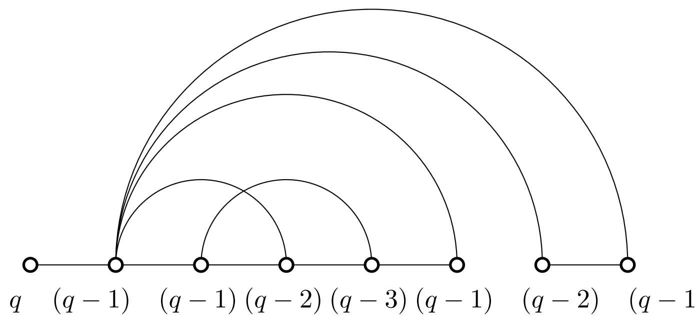



Figure 26. Available Colorings at Each Step.

See Figure 26 for a computation of the chromatic polynomial by linear factors at each vertex.

An important feature of a graph that controls the chromatic polynomial is the presence of cliques attached to a given vertex.

**Theorem 4.14.**

$$(-1)^n \chi_G(-1) = \text{ acyclic orientations of } G.$$
$$A_G = A_{G \setminus e} + A_{G/e}.$$

You can associate a polynomial to a hook diagram:

$$F_\lambda(q) = \sum_{D \text{ of shape } \lambda} q^{\#\text{ dots in } D}.$$

There is a special bipartite graph connected to a Young diagram. By considering each row and column as a vertex and an edge if there is a square in the intersection.

**Definition 4.15.**

$$\chi_\lambda := \chi_{G_\lambda}(q).$$

**Theorem 4.16.** *The number of hook diagrams $F_\lambda(1) = (-1)\chi_\lambda(-1)$ (acyclic orientations of $G_\lambda$).*