OPEN PROBLEM SESSION
ECCO 2012

(1). (Federico Ardila) For a composition $c = (c_1, \dots, c_k)$ we are interested in the **composition polynomial** $g_c(q)$, which can be given at least three definitions.
(a.) If we write $\mathbf{t^{c-1}} := t_1^{c_1-1} \cdots t_k^{c_k-1}$, where $t = (t_1, \dots, t_k)$, then

$$g_c(q) := \int_q^1 \int_q^{t_k} \cdots \int_q^{t_2} \mathbf{t^{c-1}} dt_1 \cdots dt_k.$$

(b.) Let $\beta_i = c_1 + \cdots + c_i$ for $i = 0, \dots, k$. Let $h(x) = a_0 + a_1 x + \cdots + a_k x^k$ be the polynomial of smallest degree that passes through the $k+1$ points (β_i, q^{β_i}) on the curve $y = q^x$. Here the coefficients a_i are functions of q. Then $a_k = (-1)^k g_c(q)$.
(c.) It is the volume of a combinatorially defined polytope, as explained in [1].
Some examples are:

- $g_{(1,1,1,1)}(q) = \frac{1}{24}(1-q)^4$.
- $g_{(2,2,2,2)}(q) = \frac{1}{384}(1-q)^4(1+q)^4$.
- $g_{(1,2,2)}(q) = \frac{1}{120}(1-q)^3(8+9q+3q^2)$.
- $g_{(2,2,1)}(q) = \frac{1}{120}(1-q)^3(3+9q+8q^2)$.
- $g_{(5,3)}(q) = \frac{1}{120}(1-q)^2(5+10q+15q^2+12q^3+9q^4+6q^5+3q^6)$.

and it is a fact that

$$g_c(q) = (1-q)^k f_c(q)$$

where $f_c(q)$ is a polynomial with positive coefficients. [1, Theorem 6.5]
Questions:

(I) These polynomials originally arose as volumes of polytopes; why do they also appear in the polynomial interpolation of exponential functions?
(II) Are the coefficients of $f_c(q)$ unimodal? Are they log-concave?
(III) After suitable rescaling, do the coefficients of $f_c(q)$ count nice combinatorial objects?

(2). (Criel Merino) Let

$$M_{r,d} := \{ \text{ monomials over } z_1, z_2, \dots, z_d \text{ of degree } \leq r\}.$$

A set of monomials $C_{r,d}$ of degree r over the variables $z_1, z_2, \dots, z_d$ is a *covering* set for $M_{r-1,d}$ if any monomial in $M_{r-1,d}$ is a divisor for some monomial in $C_{r,d}$. Now let

$$f_{r,d} := \text{ min size of a covering set for } M_{r-1,d}.$$

Conjecture:

(I) $f_{r,d} = \#($ aperiodic necklaces with r black beads and $d-r$ white beads).
(II) $f_{r,d} = f_{d,r}$ [Remark that this is a consequence of the previous conjecture].

(3). (Bernd Sturmfels) Let $\mathcal{F}$ be a family of non-trivial subsets of $[n]$. The collection $\mathcal{F}$ defines a family C of affine hyperplane arrangements in $\mathbb{R}^{n-1}$ as follows:

$$C = \{\sum_{i \in F} x_i = 0\}_{F \in \mathcal{F}}$$

Question: How many bounded regions does this family have? This may be intractable in general, but an answer for particular families $\mathcal{F}$ would be interesting.

Now, let P be a poset on $[n]$ and put

$$\mathcal{L}[P] = \{\text{linear extensions of } P\}.$$

Question: Determine the kernel of the map

$$\phi : \mathbb{R}[p_\pi | \pi \in \mathcal{L}[P]] \to \mathbb{R}(x_1, \dots, x_n)$$

where p_π is the probability of observing the permutation π in $\mathcal{L}[P]$ and

$$\phi(p_\pi) = \prod_{i+1}^{n} \frac{1}{x_{\pi(1)} + \cdots + x_{\pi(i)}}$$

(4). (Nantel Bergeron) The space $NCSym$ is a subspace of $k\langle\langle x_1, x_2, \dots \rangle\rangle$. For n fixed the following questions are open:

(I) Is $\langle NCSym^+_{(n)} \rangle$ finitely generated?

(II) Is the dimension of the vector space $k\langle x_1, \dots, x_n \rangle / \langle NCSym^+_{(n)} \rangle$ finite?

(III) What would be the representation theory of S_n on this quotient?

(IV) Same questions are unsolved for the space $NCQSym$.

(5). (Mauricio Velasco) Let

$$\mathcal{H}^n_d = \{I \subseteq R = k[x_1, \dots, x_n] \mid \dim_k(R/I) = d\}$$

This is the Hilbert scheme on d points in affine n-space. Now let

$$\varphi(d, n) := \sup_I \dim_k(Hom(I, R/I))$$

Question: What is $\varphi(3, n)$?

(6.) (Alejandro Morales) We denote by $\mathfrak{S}_n$ the group of permutations on $[n] = \{1, 2, \dots, n\}$. We write permutations as words $w = w_1 w_2 \cdots w_n$ where w_i is the image of w at i. We also identify each permutation w with its permutation matrix, the $n \times n$ 0-1 matrix with ones in positions (i, w_i). We think of the 1s in a permutation matrix as n non-attacking rooks on $[n] \times [n]$. Given a subset B of $[n] \times [n]$ we look at **rook placements** C of n non-attacking rooks on B.

Recall the notion of the **strong Bruhat order** $\prec$ on the symmetric group [2, Ch. 2]: if t_{ij} is the transposition that switches i and j, we have as our basic relations that $u \prec u \cdot t_{ij}$ in the strong Bruhat order when $\text{inv}(u) + 1 = \text{inv}(u \cdot t_{ij})$, and we extend by transitivity. Let $[w, w_0]$ denote the interval $\{u \mid u \succ w\}$ in the strong Bruhat order where w_0 is the largest element $n\, n-1 \dots 21$ of this order.

$$\overset{R_{35142}}{\begin{bmatrix} 0 & 0 & \underline{a_{13}} & a_{14} & a_{15} \\ 0 & 0 & a_{23} & 0 & \underline{a_{25}} \\ \underline{a_{31}} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & 0 & a_{43} & \underline{a_{44}} & a_{45} \\ a_{51} & \underline{a_{52}} & a_{53} & a_{54} & a_{55} \end{bmatrix}} \quad \overset{H_L(35142)}{\begin{bmatrix} 0 & 0 & \underline{a_{13}} & a_{14} & a_{15} \\ 0 & 0 & a_{23} & a_{24} & \underline{a_{25}} \\ \underline{a_{31}} & a_{32} & a_{33} & a_{34} & 0 \\ a_{41} & a_{42} & a_{43} & \underline{a_{44}} & 0 \\ a_{51} & \underline{a_{52}} & 0 & 0 & 0 \end{bmatrix}}$$

FIGURE 1. Matrices indicating the (i) Rothe diagram and (ii) left hull of $w = 35142$. The matrix entries $a_{i\,w_i}$ are in red.

Example 1. If $w = 3412$, then the permutations in $\mathfrak{S}_4$ that succeed w in the Bruhat order are $\{3412, 3421, 4312, 4321\}$.

In [4], Sjöstrand gave necessary and sufficient conditions for $[w, w_0]$ to be equal to the set of rook placements of a skew shape associated to w. Namely, the **left hull** $H_L(w)$ of w is the smallest skew shape that covers w. Equivalently, $H_L(w)$ is the union over non-inversions (i, j) of w of the rectangles $\{(k, \ell) \mid w_i \leq k \leq w_j, i \leq \ell \leq j\}$. See Figure 1 for an example of the left hull of a permutation.

Theorem 2 ([4, Cor. 3.3])**.** *The Bruhat interval* $[w, w_0]$ *in* $\mathfrak{S}_n$ *equals the set of rook placements in the left hull* $H_L(w)$ *of* w *if and only if* w *avoids the patterns* $1324, 24153, 31524$*, and* 426153.

A natural family of diagrams is the collection of **Rothe diagrams** of permutations, which appear in the study of Schubert calculus. The Rothe diagram R_w of a permutation w is a subset of $\{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\}$ whose cardinality is equal to the number of inversions of w; it is given by

$$R_w = \{(i, j) \mid 1 \leq i, j \leq n,\ w(i) > j,\ w^{-1}(j) > i\}.$$

See Figure 1 for some examples of Rothe diagrams. The following is a special case of two conjectures in [3, Sec. 6.].

Conjecture 3 ([3])**.** *Fix a permutation* w *in* $\mathfrak{S}_n$*. We have that the number of rook placements in the left hull* $H_L(w)$ *equals the number of rook placements in the Rothe diagram* R_w *if and only if* w *avoids the patterns* $1324, 24153, 31524$*, and* 426153.

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