1. August 24, 2015

**Algebraic topology**: take “topology” and get rid of it using combinatorics and algebra. 

Topological space $\mapsto$ combinatorial object $\mapsto$ algebra (a bunch of vector spaces with maps).

**Applications**:
- (1) Dynamical Systems (Morse Theory)
- (2) Data analysis. Topology can distinguish data sets from topologically distinct sets.

1.1. **Euclidean topology.** Working in $\mathbb{R}^n$, the distance $d(x, y) = ||x - y||$ is a metric.

**Definition 1.1.** Open set $U$ in $\mathbb{R}^n$ is a set satisfying $\forall x \in U \exists \epsilon$ s.t. $O_\epsilon(x) = \{ y \mid ||y - x|| < \epsilon \} \subset U$

1.2. **Topological Spaces.**

**Definition 1.2.** A topological space is a pair $(X, T)$ such that $X$ is a set, and $T \subseteq 2^X$ is a set of subsets of $X$ that satisfy:
- (1) $\emptyset, X \in T$.
- (2) If $A_1, \ldots, A_k \subset T$ then $\bigcap_{i=1}^k A_i \in T$. (Finite Intersection)
- (3) For any collection $\{A_i\} \subseteq T$, the union $\bigcup_{i \in I} A_i \in T$. (Arbitrary Union)

**Example 1.3.** Some sample topologies:
- (1) Discrete topology: $T = 2^X$.
- (2) Indiscrete topology: $T = \{ \emptyset, X \}$.
- (3) The induced topology on a metric space. Metric spaces have a metric which is positive-definite, symmetric and satisfies the triangle inequality. $T = \{ U \subseteq X : \forall x \in U \exists \epsilon$ s.t. $O_\epsilon(x) \subseteq U \}$.

1.3. **Topology induced by a map.** Let $(X, T_X)$ be a topological space. Let $f : X \to Y$ be a map of sets. Assume $f(X) = Y$ (unclear if necessary assumption).

Then $T_Y = \{ U \subset Y \mid f^{-1}(U) \in T_X \}$ is a topology. Notation: $T_Y = f_*(T_X)$.

1.4. **Quotient Topology.** Let $\sim$ be an equivalence relation on $X$. Consider $\pi : X \to X/\sim$.

**Definition 1.4.** Let $\pi_*(T_X)$ (using the induced notation) be the quotient topology on $Y = X/\sim$.

**Example 1.5.** Let $X = \mathbb{R}^1$. Let $x \sim y := (x - y) \in 2\pi \mathbb{Z}$. Then $Y = (X/\sim) \cong S^1$. A map to get this would be $\pi : \mathbb{R} \to S^1, \pi(\theta) = e^{i\theta}$.

**Example 1.6.** $S^n = B^n/\sim$ where $x \sim y \iff ||x|| = ||y|| = 1$. Think about folding a disk of aluminum foil over a 2-sphere, so that the edges all go to the north pole.

**Definition 1.7.** A map of topological spaces $f : X \to Y$ is continuous iff for all open $U \in T_Y$, $f^{-1}(U) \in T_X$.

2. August 26, 2015

2.1. **Review.**

- Topological space $(X, T_X)$
- Forgotten definition: Closed set is the complement of an open set.
- Induced topologies
  - by a map $f : X \to Y$.
  - by a metric $(X, d_X)$.
- by a subset $A \subset X$. $(X, T_X) \to (A, T_A)$. $T_A = \{ A \cap U \mid U \in T_X \}$.
- Continuous maps are maps where the preimage of an open set is open.
2.2. Homeomorphism.

**Definition 2.1.** Let \( f : X \to Y \) be a map of spaces; \( f \) is a homeomorphism if 1) \( f \) is a bijection, 2) \( f \) is continuous, and 3) \( f^{-1} \) is continuous.

**Remark 2.2.** If \( f \) is a bijection AND \( f(x) \) is continuous, \( f^{-1} \) is not necessarily continuous. For example, if \( f \) is the identity, and \( T_1 \) is discrete and \( T_2 \) is indiscrete.

Note that \( X \simeq Y \) is an equivalence relation. So the category of topological spaces is often defined modulo homeomorphism.

**Example 2.3.** The two realizations of \( S^n \) that we defined last class are homeomorphic.

**Example 2.4.** The open interval is homeomorphic to \( \mathbb{R}^1 \) under the tangent function.

**Example 2.5.** The open interval and the half-open interval (using the induced topology) are not homeomorphic.

2.3. Connected spaces. To prove this last example, we make two definitions:

**Definition 2.6.** A space \( X \) is *connected* if the only subsets of \( X \) that are both open and closed are \( X \) and \( \emptyset \).

A space \( X \) is *disconnected* if \( \exists U, V \) nonempty open s.t. \( X = U \cup V \) and \( U \cap V = \emptyset \).

A space is connected if and only if it is not disconnected.

**Proof.** Let \( X = (0,1) \) and \( Y = (0,1], f : X \to Y \). Take \( x = f^{-1}(1) \). Then \( X \setminus x \) should be connected and open, since it is the preimage of a connected open set. However, this is not so. Why is this true? The homeomorphism acting on a disconnection will give a disconnection of the target. \( \square \)

**Definition 2.7.** A space \( X \) is *path-connected* if given any two points \( x, y \in X \) there is a continuous map \( [0,1] \to X \) with \( f(0) = x \) and \( f(1) = y \).

**Lemma 2.8.** \( X \) path-connected implies \( X \) connected.

The converse is not true but requires some pathological behavior.

There is an equivalence relation \( \sim \) on \( X \) setting \( x \sim y \iff \exists \) continuous path from \( x \) to \( y \).

**Definition 2.9.** (Path-connected components of \( X \)) := \( X/\sim \).

**Exercise 2.10.** Let \( X \approx \) the 2-sphere \( S^2 \), and \( Y \) the 2-torus \( T^2 \).

Prove these are not homeomorphic. Cut a circle out of the torus, map to the sphere. The result should be (path-)connected; however, that’s impossible.

**Definition 2.11.** \( X \) is Hausdorff means \( x \neq y \in X \) then \( \exists \) open \( U \) containing \( x \) and open \( V \) containing \( y \) that are disjoint.

**Example 2.12.** Non-Hausdorff space: Take \( X \) and \( Y \) two copies of \( \mathbb{R}^1 \). Glue them together except at the origin; i.e. \( X \sqcup Y/\sim \) where \( \sim := x \sim y \iff x = y \neq 0 \).
3.1. **Review.**

**Theorem 3.1.** If $M$ is a compact 2-dimensional manifold without boundary then:
- If $M$ is orientable, $M = H(g) = \#^g \mathbb{P}^2$.
- If $M$ is nonorientable, $M = M(g) = \#^g \mathbb{R}\mathbb{P}^2$.

Terminology: $g$ is the genus of the surface = maximal number of closed paths one can cut out without disconnecting.
Note: No higher-dimensional analogue exists (2-dimensions is trivial). Note: $H(0) = S^2$ by definition.

3.2. **Gluing diagrams.**

**Definition 3.2.** Edges are “decorated” with letter:
- $a$ means that the orientation is clockwise.
- $a^{-1}$ means that the orientation is counterclockwise.

For each letter the edges are glued according to the orientation.

**Example 3.3.** Below are some examples of gluing diagrams:

\[
\begin{array}{ccc}
  & a & \\
\begin{array}{c}
  b^{-1}
  \\
  b
  \\
\end{array}
  & a^{-1}
  & \\
\begin{array}{c}
  b
  \\
  b^{-1}
  \\
\end{array}
  \\
  & a
\end{array}
\]

\[
\begin{array}{ccc}
\cong S^2 & \cong T^2 & \cong \text{Klein Bottle}
\end{array}
\]

$w = aba^{-1}cd\cdots gf$ is a word describing the circumference of a polygon. Simple properties:
1. Cyclic permutation preserves the homeomorphism class.
2. Inserting $aa^{-1} \cong$ connected sum with a sphere; therefore, it preserves the homeomorphism class.
3. Concatenating two words amounts to connected sum of the corresponding manifolds (really, concatenating the inverse of one, but the inverse is isomorphic to itself).

**Example 3.4.** Show that the Klein bottle is homeomorphic to $\mathbb{R}\mathbb{P}^2 \# \mathbb{R}\mathbb{P}^2$.

Flip

\[
\begin{array}{ccc}
\begin{array}{c}
  a
  \\
  b^{-1}
  \\
\end{array}
  & d
  & \\
\begin{array}{c}
  d
  \\
  d
  \\
\end{array}
  \\
  & a
\end{array}
\]

Proof. $aba^{-1}b \cong abdd^{-1}a^{-1}b = (abd)(d^{-1}a^{-1}b) = (abd)(b^{-1}ad) = (daad) \cong \mathbb{R}\mathbb{P}^2 \# \mathbb{R}\mathbb{P}^2$. \hfill \Box

The same logic would apply to prove that $T^2 \# \mathbb{R}\mathbb{P}^2 \cong \#^3 \mathbb{R}\mathbb{P}^2$; manipulating the perimeter words eventually obtains the result.
3.3. **Triangulations.** The topology of any 2-d manifold can be determined by a collection of triangles and how they are glued together.

**Definition 3.5.** A triangulation of a 2-d manifold $M$ is a collection of $T_i \subset M$ s.t. if $T_i \cap T_j \neq \emptyset$ then either $T_i \cap T_j =$ one edge of each triangle or $T_i = T_j =$ a single point which is a vertex of each triangle.

**Theorem 3.6.** *Every compact 2-dim manifold has triangulations.*

4. **September 4, 2015**

4.1. **Review.** Triangulations

**Example 4.1.** The 2-sphere is relatively easy to triangulate. Take three circumferences and their points of intersection.

The resulting complex is an octahedron.

**Example 4.2.** The torus is a bit harder to triangulate. The triangulation on the right fails since the two gray triangles have two vertices in common but no edge.

**Definition 4.3.** An Euler characteristic of a triangulation is given by

$$\chi(T) = V - E + F$$

**Theorem 4.4.** *The Euler characteristic of a triangulation depends only on the homeomorphism class of the manifold.*

**Proposition 4.5.** $\chi(H(g)) = 2 - 2g$ and $\chi(M(g)) = 2 - g$.

*Proof.* The proof will only come much later.
4.2. (Geometric) Simplicial Complexes.

**Definition 4.6.** Let \( u_0, \ldots, u_k \in \mathbb{R}^d \). An affine combination of \( u_0, \ldots, u_k \) is
\[
x = \sum_{i=0}^{k} \lambda_i u_i; \quad \lambda_i \in \mathbb{R}
\]
with the condition \( \sum_{i=0}^{k} \lambda_i = 1 \).

The set of affine combinations of two points is a line. The set of affine combinations of \( 3 \) (linearly independent) points is a 2-plane.

**Definition 4.7.** The affine hull of \( u_0, \ldots, u_k \) is the set of all possible affine combinations.

**Definition 4.8.** The points \( u_0, \ldots, u_k \) are affinely independent if
\[
\sum_{i=0}^{k} \lambda_i u_i = \sum_{i=0}^{k} \mu_i u_i \iff \lambda = \mu \in \mathbb{R}^{k+1}.
\]

**Remark 4.9.** The points \( u_0, \ldots, u_k \) are affinity independent if and only if \( v_i = u_i - u_0 \) for \( i = 1, \ldots, k \) are linearly independent.

**Corollary 4.10.** There are at most \( (d+1) \) affinely independent points in \( \mathbb{R}^d \).

If \( k \leq d + 1 \) then the set of points \( \{u_0, \ldots, u_k\} \subset \mathbb{R}^{d(k+1)} \) that are dependent has zero measure (in the standard measure on that space).

**Definition 4.11.** A convex combination of \( u_0, \ldots, u_k \) is a point \( \sum_{i=0}^{k} \lambda_i u_i \), where \( \sum_{i=0}^{k} \lambda_i = 1 \) and \( \lambda_i \geq 0 \) for all \( i \).

**Definition 4.12.** A convex hull of \( u_0, \ldots, u_k \) is
\[
\text{conv}\{u_0, \ldots, u_k\} = \left\{ \sum_{i=0}^{k} \lambda_i u_i : \sum_{i=0}^{k} \lambda_i = 1, \lambda_i \geq 0 \right\}
\]

**Example 4.13.** The convex hull of two points is a line segment.
The convex hull of three points is a triangle.
This assumes the points are not affinely independent.

**Definition 4.14.** Assume \( u_0, \ldots, u_k \in \mathbb{R}^d \) are affinely independent.

\( S = \text{conv}\{u_0, \ldots, u_k\} \) is called a simplex. Define the dimension of \( S \) to be \( k \).

The empty simplex is a simplex by convention, with dimension \(-1\).

**Definition 4.15.** A face of a simplex \( S = \text{conv}\{u_0, \ldots, u_k\} \) is a simplex \( T = \text{conv}\{u_{\alpha_0}, \ldots, u_{\alpha_k}\} \) where \( \alpha \subseteq \{0, 1, \ldots, k\} \).

**Exercise 4.16.** For all \( x \in S \), \( x \) is in the interior of exactly one face of \( S \).

For this we need to define the boundary \( \text{bd}(S) = \{\cup_i T_i | T_i = \text{conv}\{U_j | j \neq i\}\} \). Then the interior of the face is \( \text{int}(S) = S \setminus \text{bd}(S) \).

**Proof.** Let \( x \in S \). This implies that there exist \( \lambda_0, \ldots, \lambda_k \) such that \( x = \sum_{i=0}^{k} \lambda_i u_i \). Then \( T \) is unique face of \( S \) such that \( x \in \text{int}(T) \) and \( \alpha = \text{supp}(\lambda) = \{i \mid \lambda_i > 0\} \).

**Definition 4.17.** A (geometric) simplicial complex is a collection \( K = \{S_0\} \) of simplices, such that

1. If \( T \subseteq S, S \in K \Rightarrow T \in K \).
2. If \( S_1, S_2 \in K \) then \( S_1 \cap S_2 \) is a face of both \( S_1 \) and \( S_2 \), where we consider the empty set to be a face of every simplex.
The dimension of $K$ is defined as the maximal dimension of its faces. The underlying space $|K| = \bigcup_{S \in K} S$ is the underlying space with the induced topology.

**Definition 4.18.** The triangulation of a topological space $X$ is a pair $(K, f : K \to X)$ where $K$ is a geometric simplicial complex and $f : K \to X$ is a homeomorphism.

**Sales pitch:** When we have a triangulation, everything about the topology of $X$ is encoded in the combinatorics of $K$.

**Definition 4.19.** An abstract simplicial complex will be defined next class.

### 5. September 9, 2015

**Definition 5.1.** Let $V$ be a set, then a collection of subsets $A \subset 2^V$ will be called an abstract simplicial complex if it is closed downward, i.e. if $\sigma \in A$ and $\tau \subset \sigma$ then $\tau \in A$.

**Example 5.2.** The following are abstract simplicial complexes: $A = \emptyset$ – no subsets; $A = \{\emptyset\}$ – not empty: it contains the set $\emptyset$. $A = \{\emptyset, \{1\}\}$ with the ambient set $V = \{1\}$.

An example of a non-simplicial complex is $A = \{\{1\}, \{1, 2\}\}$ – this is not simplicial because even though $\{2\} \subset \{1, 2\}$, we do not have $\{2\} \in A$.

**Remark 5.3.** For any geometric simplicial complex there exists a unique abstract simplicial complex such that $K = \{S(\alpha) = \text{conv}\{p_i\}_{i \in \alpha}\}$

$V$ is defined as the set of 0-dimensional simplices.

Then $A = \{\alpha \in 2^V \mid \exists S \in K : S = \text{conv}\{p_i\}_{i \in \alpha}\}$.

**Example 5.4.** Consider the following geometric simplicial complex.

![Diagram](image)

Here $V = \{1, 2, 3, 4\}$, $A \subset 2^V$ is given by $A = \text{the subsets of } \{1, 2, 3\}$ and $\{2, 3, 4\}$.

**Definition 5.5.** Such an abstract simplicial complex is called the vertex scheme.

**Remark 5.6.** If $p_i \in \mathbb{R}^N$. Denote $S(\alpha) = \text{conv}\{p_i\}_{i \in \alpha}$.

<table>
<thead>
<tr>
<th><strong>Abstract</strong></th>
<th><strong>Geometric</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta \subseteq \alpha$</td>
<td>$T(\beta) \subseteq S(\alpha)$</td>
</tr>
<tr>
<td>$V$</td>
<td>vertices of $K$</td>
</tr>
<tr>
<td>$\dim S = \text{card}(\alpha) - 1$</td>
<td>$\dim S = d$</td>
</tr>
<tr>
<td>$\dim A = \max_{\alpha \in A}(\dim \alpha)$</td>
<td>$\dim A = \max_{S \in A} \dim S$</td>
</tr>
</tbody>
</table>

**Table 1.** Analogous Properties of Abstract and Geometric Simplicial Complexes
Theorem 5.7 (Geometric Realization Theorem). Let $A$ be an abstract simplicial complex of dim $A = d$ then there exists a geometric realization in $(2d + 1)$-dimensional space.

Remark 5.8. $2d + 1$ is a tight condition for all $d$. There exist examples of complexes not realizable in dimension $2d$. For example with $d = 2$, the complete graph $K_5$ is a 1-dimensional complex; since it is nonplanar, it cannot be embedded in dimension $2d = 2$ without self-intersections. The rules of geometric simplicial complexes however demand that all intersections of faces are themselves faces of the complex.

Lemma 5.9. Any $(m + 1)$ distinct points

$$\gamma(t_0), \gamma(t_1), \ldots, \gamma(t_m)$$

where $\gamma(t) = (t, t^2, \ldots, t^m)$

are affinely independent if and only if $t_i \neq t_j$.

Proof. The determinant given by:

$$\det \begin{pmatrix} 1 & t_0 & t_0^2 & \cdots & t_0^m \\ 1 & t_1 & t_1^2 & \cdots & t_1^m \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & t_m & t_m^2 & \cdots & t_m^m \end{pmatrix} = \prod_{0 \leq i < j \leq m} (t_j - t_i)$$

is the Vandermonde determinant which is only zero if two $t$-values are the same. □

Corollary 5.10. For every finite set $V$, there exists a map $p : V \to \mathbb{R}^{2d+1}$ such that any $k \leq 2d + 2$ are affinely independent.

Proof. $A \subset 2^V$ is an abstract simplicial complex with $V =$ the set of vertices and dim $A = d$ given by the maximal cardinality of a face of $A$.

For each $r \in V$, we have $p_r \in \mathbb{R}^{2d+1}$ such that any $2d + 2$ points are affinely independent.

We can define $\forall \alpha \in A$:

$$S(\alpha) := \text{conv}\{p_r\}_{r \in \alpha}.$$ 

This is always a simplex because the points are affinely independent.

Now we need to confirm the simplicial complex axioms.

(1) $S$ is a simplex.
(2) $T \subseteq S, S \in K \implies T \in K$. (True because if $\alpha \in A, \beta \subseteq \alpha \implies \beta \in A.$)
(3) $S_1, S_2 \in K$, then $S_1 \cap S_2$ is either empty or a face of each.

The first two are trivial. Proving (2), let $S_1 = S(\alpha_1), S_2 = S(\alpha_2)$.

$$\text{card}(\alpha_1 \cup \alpha_2) = \text{card}(\alpha_1) + \text{card}(\alpha_2) - \text{card}(\alpha_1 \cap \alpha_2)$$

$$\implies \text{card}(\alpha_1 \cup \alpha_2) \leq (d_1 + 1) + (d_2 + 1) - 2d + 2$$

Thus conclude that the vertices are affinely independent. We need to show: $X \in S_1 \cap S_2 \implies X$ is a face of $S_1$. Recall that a convex combination of affinely independent points has a unique formulation. Thus there is a specific $\beta_1 \subseteq \alpha_1$ and $\beta_2 \subseteq \alpha_2$, such that $X = \sum y_r p_r$, and $\beta_1 = \beta_2 = \text{supp } y$; in particular $\beta_1 = \beta_2 = \alpha_1 \cap \alpha_2$. □


The geometric realization theorem sets up a correspondence between abstract simplicial complexes and geometric simplicial complexes.

Let $K, L$ be two (geometric) simplicial complexes.
Definition 6.1 (1). A PL-map \( f : K \to L \) is a map defined on each simplex of \( K \) as:
\[
f \left( \sum_{i=0}^{k} \alpha_i U_i \right) = \sum_{i=0}^{k} \alpha_i f(U_i)
\]
PL stands for piecewise linear.

Note that the map is uniquely specified by the values on the vertices.

Definition 6.2 (1*). Let \( A \subset 2^V, B \subset 2^U \) be two abstract simplicial complexes.
A simplicial map is a map \( m : A \to B \) that satisfies \( \forall \sigma \in A, \)
\[
\sigma = (i_0, i_1, \ldots, i_k) = \bigcup_{j=0}^{k} \{i_j\} \implies m(\sigma) = (i_0, i_1, \ldots, i_k) = \bigcup_{j=0}^{k} m(\{i_j\}).
\]

Definition 6.3 (1**). Let \( A, B \) be a simplicial complex with \( V = \text{vert}(A), U = \text{vert}(B), \)
then a map \( m_0 : V \to U \) is simplicial if \( \forall \sigma \in A, \)
\[
\bigcup_{V \in \sigma} m_0(V) \in B
\]

Remark 6.4. The following diagram commutes:
\[
\begin{align*}
K \xrightarrow{p_L} & \text{vertex scheme } A_K \\
\downarrow & & \downarrow m \text{ simplicial map} \\
L \xrightarrow{m} & \text{vertex scheme } A_L
\end{align*}
\]

Definition 6.5 (2). A PL map is a PL homeomorphism if it is a bijection on each simplex.

Definition 6.6 (2*). A simplicial map is a simplicial complex isomorphism iff \( m_0 \) is a bijection.

Example 6.7. The image on the left and the right are not isomorphic as simplicial complexes but a subdivision of the left complex – given by the central complex – isomorphic to the one at right.

![Figure 1. Subdivision of the Simplicial Complex Yields Isomorphism](image)

Definition 6.8. A subdivision of a geometric complex adds in faces as in Example 6.7

Conjecture 6.9 (This was FALSE!). Two compact manifolds are isomorphic if and only if their triangulations have isomorphic schemata after a finite number of subdivisions.

Theorem 6.10. This conjecture holds for \( \dim M \leq 3 \).
Definition 6.11. Let $A \subset 2^V$ be an abstract simplicial complex, then $Sd(A)$, the barycentric subdivision, is a simplicial complex $Sd(A) \subset 2^{A \setminus \emptyset}$ where $V \subset A$ is in $Sd(A) \iff V = \{\sigma_0, \ldots, \sigma_k\}$ such that $\sigma_0 \subsetneq \sigma_1 \subsetneq \cdots \subsetneq \sigma_k$.

Example 6.12. We perform barycentric subdivision of the 1-simplex and 2-simplex.

In general, if $K = \{S_a\}$, for each $S = \{u_0, \ldots, u_k\}$ a simplex, introduce a new vertex

$$U_S = \frac{1}{k+1} \sum_{i=0}^k U_i$$

and define simplices according to the same rule as in the abstract simplicial complex.

Exercise 6.13 (Homework Qs).

1. Why is a $\Delta$-complex not a triangulation?
2. Why is a triangulation not a $\Delta$-complex?
3. What is the role of the vertex ordering in the $\Delta$-complex induced by a triangulation?

7. September 18, 2015

No class on September 14, notes from Sep 16 to be posted later.

7.1. Simplicial Homology of $\Delta$-complexes. Let $G$ be an abelian group.

Definition 7.1. The chain group

$$\Delta_n(X; G) = \{ \sum_{\sigma \dim n} a_\sigma \sigma \}$$

Without specified group, take

$$\Delta_n(X) = \Delta_n(X; \mathbb{Z}).$$

The boundary homomorphism maps:

$$\partial_n : \Delta_n(X; G) \rightarrow \Delta_{n-1}(X; G)$$

$$\partial_n(\sigma) = \sum_{i=0}^n (-1)^i \sigma|_{v_0, v_i, \ldots, v_n}$$

The homology of a complex is the direct sum of the graded homology groups:

$$H_* (X; G) = \bigoplus_{n=0}^\infty H_n(X; G).$$

Now we return to the example of $\mathbb{R}P^2$: 
Example 7.2. \( H_*(\mathbb{R}P^2, \mathbb{Z}/2\mathbb{Z}) \). We use the \( \Delta \)-complex in Figure 2 to compute the homology.

![Figure 2. \( \Delta \)-Complex for \( \mathbb{R}P^2 \)](image)

The chain groups at each step are given in this sequence:

\[
0 \xleftarrow{} \Delta_0 \xleftarrow{} \Delta_1 \xleftarrow{} \Delta_2 \xleftarrow{} 0
\]

\[
0 \xleftarrow{} (\mathbb{Z}_2)^2 \xleftarrow{} (\mathbb{Z}_2)^3 \xleftarrow{} (\mathbb{Z}_2)^2 \xleftarrow{} 0
\]

We can compute each kernel and image in order to find homology:

- \( \ker(\partial_2) = \langle U + L \rangle \), \( \text{Im}(\partial_3) = \langle 0 \rangle \)
- \( \ker(\partial_1) = \langle a + b, c \rangle \), \( \text{Im}(\partial_2) = \langle a + b + c \rangle \)
- \( \ker(\partial_0) = \langle v, w \rangle \), \( \text{Im}(\partial_1) = \langle w - v \rangle \)

Therefore \( H_*(\mathbb{R}P^2, \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & * = 0, 1, 2 \\ 0 & \text{else.} \end{cases} \)

Remark 7.3. If \( A \subset 2^V \) is an abstract simplicial complex, then

\[
C_n(A; G) = \{ \sum_{|\sigma|=n+1} a_\sigma \sigma | a_\sigma \in G \}
\]

The boundary map and the homology groups are defined as before.

Moreover if \( X = |A| \), the geometric realization of \( A \), then \( H_*(A, G) \cong H_\Delta(|A|, G) \); the abstract homology is the same as the \( \Delta \)-complex homology.

7.2. Singular Homology.

Definition 7.4. A singular \( n \)-simplex in a topological space \( X \) is a continuous map \( \sigma : \Delta^n \to X \).

Definition 7.5. Singular chains (with coefficients in \( G \))

\[
C_n(X; G) = \{ \sum_{\sigma \in I} a_\sigma \sigma | a_\sigma \in G \}
\]

(only finitely many \( a_\sigma \) are nonzero; i.e. \( I \) finite).

The boundary homomorphism

\[
\partial_n : C_n(X; G) \to C_{n-1}(X; G) \quad \sigma \mapsto \sum_{i=0}^n (-1)^i \sigma|_{[v_0, \ldots, \hat{v}_i, \ldots, v_n]}
\]
Definition 7.6. \(H^{\text{Sing}}(X; G) \cong \ker(\partial_n)/\text{Im}(\partial_{n+1}).\)

Theorem 7.7. \(H^\Delta_*(X; G) \cong H^\text{Sing}_*(X; G).\)

Question 7.8. Why is this nicer to have?

The singular homology has nice functorial properties. For example, \(f : X \to Y\) continuous induces \(f_* : H_n(X) \to H_n(Y)\) group homomorphism.

For \(a \in C_n(X; G)\) where \(a = \sum \sigma a\sigma\); then \(f_\sharp a = \sum \sigma a_\sharp f_\sharp(\sigma).\)

Remark 7.9. The maps commute: \(\partial_n f_\sharp(a) = f_\sharp(\partial_n a).\)

Exercise 7.10 (Homework). Prove that the map \(f_\sharp : C_n(X) \to C_n(Y)\) is a group homomorphism that “extends” to \(f_* : H_n(X) \to H_n(Y)\) via \(f_\sharp(a + \text{Im} \partial_{n+1}) := f_\sharp a + \text{Im} \partial_{n+1}.\)

Proposition 7.11. If \(X \cong Y\) homeomorphic then if \(f : X \to Y\) is a homeomorphism then \(f_* : H_*(X) \to H_*(Y)\) is a group isomorphism.

8. September 21, 2015

8.1. Last few classes.

- Simplicial homology \(H^\Delta_*(X; G)\)
- Singular homology \(H^\text{Sing}_*(X; G)\)

The last theorem we discussed in class:

Proposition 8.1. If \(X \cong Y\) homeomorphic then if \(f : X \to Y\) is a homeomorphism then \(f_* : H_*(X) \to H_*(Y)\) is a group isomorphism.

Definition 8.2. A graded abelian group is \(C = \bigoplus_{i \in \mathbb{Z}} C_i\) where \(C_i\) are abelian groups.

Definition 8.3. A chain complex is a graded abelian group with group homomorphisms \(\partial_i : C_i \to C_{i-1}\) such that \(\partial_{i-1} \circ \partial_i = 0.\)

\(Z_i(C) = \ker(\partial_i : C_i \to C_{i-1})\) are cycles.

\(B_i(C) = \text{Im}(\partial_{i+1} : C_{i+1} \to C_i)\) are boundaries.

\(H_i(C) = Z_i(C)/B_i(C).\)

Let \(C_* = \bigoplus_i C_i\) and \(D_* = \bigoplus_i D_i\) be chain complexes.

Definition 8.4. A chain map is a collection of group homomorphisms \(f_i : C_i \to D_i\) such that the following diagram commutes:

\[
\begin{array}{ccc}
C_i & \xrightarrow{f_i} & D_i \\
\downarrow{\partial_i} & & \downarrow{\partial_i} \\
C_{i-1} & \xrightarrow{f_{i-1}} & D_{i-1}
\end{array}
\]

Lemma 8.5. A chain map induces a group homomorphism \(f_* : H_i(C) \to H_i(D).\)

Proof. \(H_i(C) = \ker(\partial_i)/\text{Im}(\partial_{i+1}).\) Let \(c \in C_i\) be a cycle such that \(\partial c = 0.\) Notation: \([c] := c + \text{Im} \partial_{i+1} \in H_i(C).\)

Define \(f_*([c]) = [f(c)] = f(c) + \text{Im} \partial_{i+1} \in H_i(D).\) We need to show that:
\[
\partial f(c) = 0. \text{ [This follows from } \partial f = f \partial.] \\
\text{(2) If } \tilde{c} = c + \partial a \text{ then } [f(\tilde{c})] = [f(c)]. \text{ [This follows by } f \text{ being a group homomorphism.]} \\
\]

\textbf{Corollary 8.6.} If \( g : X \to Y \) is a continuous map of topological spaces, then \( g_* : C_i(X) \to C_i(Y) \)
induces a group homomorphism
\[
g_* : H_i(X; G) \to H_i(Y; G)
\]

\textit{Proof.} \( g_* : C_i(X; G) \to C_i(Y; G) \) is a chain map. \( \square \)

\textbf{Lemma 8.7.} If \( f : C_* \to D_* \) is a chain group isomorphism (i.e. \( f_i : C_i \to D_i \) are group isomorphisms) and \( \partial f_i = f_{i-1} \partial_i \), then
\[
f_* : H_i(C) \to H_i(D)
\]
is a group isomorphism.

\textit{Proof.} Chase some diagrams. \( \square \)

\textbf{Corollary 8.8.} If \( g : X \to Y \) is a homeomorphism, then \( g_* : H_i(X; G) \to H_i(Y; G) \) is a group isomorphism.

\textit{Proof.} \( g_* : C_i(X; G) \to C_i(Y; G) \) such that \( \forall \sigma : \Delta^i \to X \)
\[
g_*(\sigma) = g \circ \sigma \text{ is a chain map. Notice that it has a chain map inverse.}
\]
Note that \( (g_*^{-1}) \circ (g_*) = Id_{C_i(X; G)}. \) Use the Lemma. \( \square \)

\textbf{Remark 8.9.} The converse is not true. \( H_*(S^1 \times \mathbb{R}^1) = H_*(S^1) \) but the spaces are not homeomorphic.

Another simple lemma:

\textbf{Lemma 8.10.} Let \( C_* \) be a chain complex such that \( C_i = \bigoplus_{\alpha} C^\alpha_i \) and \( \partial_i C^\alpha_i \subseteq C^\alpha_{i-1}. \)

Then \( H_i(C) = \bigoplus_{\alpha} H_i(C^\alpha). \)

\textbf{Corollary 8.11.} If \( X = \bigsqcup_{\alpha} X_\alpha \) where \( X_\alpha \) are its path-connected components then
\[
H_i(X) = \bigoplus_{\alpha} H_i(X_\alpha).
\]

\textit{Proof.} Need to show that
\[
C_i(X; G) = \bigoplus_{\alpha} C_i(X_\alpha; G)
\]
and \( \partial C_i(X_\alpha; G) \subseteq C_{i-1}(X_\alpha; G). \)

\[
C_i(X; G) = \{ \sum_{\sigma} a_\sigma \sigma \mid a_\sigma \in G, \sigma : \Delta^i \to X \}
\]
For each \( \sigma : \Delta^i \to X \), observe that \( \sigma(\Delta^i) \) must be path-connected thus lie in one of these \( X_\alpha \) thus
\[
C_i(X; G) \cong C_i(X_\alpha; G).
\]
Note that if \( \sigma : \Delta^i \to X_\alpha \) then \( \partial \sigma \in C_{i-1}(X_\alpha; G). \) \( \square \)

\textbf{Definition 8.12.} A chain complex is called an \textit{exact sequence} if the homology is trivial.

\textbf{Lemma 8.13.} If \( 0 \leftarrow A \leftarrow B \leftarrow C \) is an exact sequence then \( A \cong B/C. \)

**Corollary 9.1.** If $X$ is path connected then $H_0(X; G) \cong G$.

**Proof.** $H_0(X; G) \cong C_0(X; G)/\text{Im}(\partial_1 : C_1(X; G) \to C_0(X; G))$.

Define $\epsilon$ such that

$$G \xleftarrow{\epsilon} C_0(X; G) \xrightarrow{\partial_0} C_1(X; G)$$

by sending $a \in C_0(X; G)$ where $a = \sum_\sigma a_\sigma \sigma$ where $\sigma$ is a point, then $\epsilon(a) = \sum_\sigma a_\sigma$; i.e. add up all coefficients from the group. We claim that:

$$0 \leftarrow G \xleftarrow{\epsilon} C_0(X; G) \xrightarrow{\partial_0} C_1(X; G)$$

is an exact sequence. The proof of the corollary follows from this claim by Lemma 8.13.

- $\text{Im}(\epsilon) = G$. Obvious.
- $\text{Im}(\partial_1) \subseteq \ker(\epsilon)$. Let $a = \partial_1(b)$ then $\epsilon(a) = \epsilon(\partial_1(b))$ where $b = \sum_\sigma b_\sigma \sigma$ and $\sigma \in C_1$ are one-dimensional simplices. $\epsilon(a) = \epsilon(\partial_1(b)) = \epsilon(\partial_1(\sum_\sigma b_\sigma \sigma)) = \sum_\sigma b_\sigma \epsilon(\partial_1(\sigma))$ where $\sigma : [p_0, p_1] \to X$ are one-dimensional simplices.
- $\ker(\epsilon) \subseteq \text{Im}(\partial_1)$. Assume $a \in C_0(X; G)$ and $\epsilon(a) = 0$. Want to Show: $\exists b$ such that $a = \partial_1 b$.

Note that $a = \sum_\sigma a_\sigma \sigma$ where $\sigma$ is a zero-dimensional simplex, i.e. a point.

Pick any $x_0$ then there exists a path from $x_0$ to each $x_i$ corresponding to $\sigma$ since $X$ is path-connected. Indeed for each $\sigma$ there exists $p_\sigma : [0, 1] \to X$ with $p_\sigma(0) = x_0$ and $p_\sigma(1) = x_\sigma$. Define $b = \sum_\sigma a_\sigma p_\sigma$ Then

$$\partial_1(b) = \partial_1(\sum_\sigma a_\sigma p_\sigma) = \sum_\sigma a_\sigma (p_\sigma(1) - p_\sigma(0)) = \sum_\sigma a_\sigma \partial_1(p_\sigma)$$

Therefore $a \in \text{Im} \partial_1$. \hfill $\square$

10. September 25, 2015

10.1. **Last class.** We began the Mayer-Vietoris sequence. Short exact sequence $\implies$ long exact sequence.

![Figure 3. Union of topological spaces](image)

Consider the union of spaces in Figure 3. It has a short exact sequence:

$$0 \to C_k(X_1 \cap X_2) \to C_k(X_1) \oplus C_k(X_2) \to C_k(X_1 \cup X_2) \to 0.$$  

**Question 10.1.** If we understand $H_*(X_1)$ and $H_*(X_1 \cap X_2)$, what is $H_*(X_1 \cup X_2)$?

More generally, consider the commutative diagram of short exact sequences given below.
Lemma 10.2. There is a group homomorphism (connecting homomorphism) $\delta : H(C_k) \to H(C_{k-1})$ such that $\delta([c]) = [a]$, where $a$ is defined as above.

Proof. Need to show:

(1) $\partial a = 0$.
(2) Independent of choice of $b$.

For (1), we see that $i(\partial a) = \partial(ia) = \partial \partial b$ thus $\partial a = 0$ by injectivity of $i$.

For (2), assume a choice of $\tilde{b}$ such that $j \tilde{b} = c$. $\tilde{a} = i^{-1}(\partial \tilde{b})$ Wanted: $[\tilde{a} - a] = 0$. This means $\tilde{a} - a = \partial a'$. $i(\tilde{a} - a) = \partial b - \partial b = \partial (b - b)$. Simply set $a'$ to be $i^{-1}(\tilde{b} - b)$ and the result has $\partial a' = \tilde{a} - a$ by injectivity. □

Theorem 10.3 (Short $\to$ long). Let $0 \to A_* \to B_* \to C_* \to 0$ be an exact sequence of chain complexes. Then, there is a long exact sequence:

Proof. Remark:

$$i_*[a] = [ia] \quad j_*[b] = [jb]$$

Need to prove:

$$\text{Im } i_* \subseteq \text{ker } j_* \quad \text{ker } j_* \subseteq \text{Im } i_*$$
$$\text{Im } j_* \subseteq \text{ker } \partial \quad \text{ker } \partial \subseteq \text{Im } j_*$$
$$\text{Im } \partial \subseteq \text{ker } i_* \quad \text{ker } i_* \subseteq \text{Im } \partial$$

The left-hand containments prove that we have a chain complex, while the right-hand containments prove that it is exact. Diagram-chasing ensues. □
11. September 28, 2015

11.1. Last class. Let $0 \to A_* \to B_* \to C_* \to 0$ be a short exact sequence of chain complexes. Then there is a theorem:

**Theorem 11.1.** There is a long exact sequence:

$$
\begin{align*}
H_{i+1}(A) & \overset{i_*}{\to} H_{i+1}(B) \overset{j_*}{\to} H_{i+1}(C) \\
\delta & \notag \\
H_i(A) & \overset{i_*}{\to} H_i(B) \overset{j_*}{\to} H_i(C) \\
\delta & \notag \\
& \cdots
\end{align*}
$$

What are exact sequences good for?

**Example 11.2.** In the case of Mayer-Vietoris, $A_k = C^\text{sing}_k(X_1 \cap X_2)$, $B_k = C^\text{sing}_k(X_1) \oplus C^\text{sing}_k(X_2)$, and $C_k = C^\text{sing}_k(X_1 \cup X_2)$.

If $H_k(X_1) = H_k(X_2) = 0$ for $k > 0$, then $H_k(X_1) \oplus H_k(X_2) = 0 \oplus 0 = 0$ for $k > 0$. So the exact sequence, i.e. the Mayer-Vietoris sequence tells us that

$$
0 \to H_k(X_1 \cup X_2) \overset{\delta}{\to} H_{k-1}(X_1 \cap X_2) \to 0
$$

should be exact. In particular these groups are isomorphic.

**Example 11.3.** Again we refer to Figure 3 from earlier.

**Figure 4.** Chains in the Mayer-Vietoris Sequence

*What does the map $\delta : H_1(X_1 \cup X_2) \to H_0(X_1 \cap X_2)$ do?*

It maps a pair of 1-chains from $C^\text{sing}_1(X_1) \oplus C^\text{sing}_1(X_2)$ via $C^\text{sing}_1(X_1 \cup X_2) \to C^\text{sing}_0(X_1 \cup X_2) \to C^\text{sing}_0(X_1) \oplus C^\text{sing}_0(X_2)$ a pair of 0-chains. For a 1-chain to survive the homology functor it needs to have a single vertex. In other words the singular chain $\sigma$ has $\sigma(0) = \sigma(1)$. Since $j(\alpha_1 \oplus \alpha_2) = \alpha_1 - \alpha_2$, we have $j(\alpha_1 \oplus \alpha_2) = \sigma$. This means $\partial \alpha_1 = [p_2] - [p_1] = \partial \alpha_2$. This means that $i^{-1} \delta([\sigma]) = [p_2] - [p_1]$. This specifies the value of $\delta([\sigma]) = [\beta]$.

**Example 11.4** (Triangualtion of a sphere). Let $X_1$ and $X_2$ be cones over the same triangle. Their intersection is a triangle. A sphere has nonzero $H_2$; here its generator would be $[\sigma]$ a signed sum of the six triangles. The map $\delta$ goes to the equator given by the intersection triangle.
11.2. **Homotopy equivalence.** “I hid the truth from you.” Recall: $X \cong Y$ implies $H_*(X) = H_*(Y)$.

More generally:

**Definition 11.5.** Two continuous maps $f, g : X \to Y$ are called *homotopic* if there exists continuous functions $F : X \times [0, 1] \to Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$. The function $F : X \times [0, 1] \to Y$ is called a *homotopy*.

**Example 11.6.** If $f, g$ are functions from $\mathbb{R} \to \mathbb{R}$ then a homotopy is a 2-d surface in $\mathbb{R}^3$ as pictured in Figure 5.

![Figure 5. Homotopy from $f$ to $g$.](image)

Notation: $f \sim g$ means $f$ is homotopic to $g$.

**Lemma 11.7.** Homotopy is an equivalence relation on continuous maps. In particular, $f \sim f, f \sim g \implies g \sim f$, and $f \sim g, g \sim h \implies f \sim h$.

**Definition 11.8.** $f : X \to X$ is *null-homotopic* if $f$ is homotopic to $id_X$ i.e. $f \sim id_X$.

**Definition 11.9.** Let $A \subseteq X$ be a subspace. $A$ is called a *deformation retract* of $X$ if it has a deformation retraction, a homotopy from $id_X$ to a map sending $X \to A$ which is the identity on $A$.

**Example 11.10.** Take $X$ to be the cylinder $x^2 + y^2 = 1, 0 \leq z \leq 1$ in $\mathbb{R}^3$ and map $(x, y, z) \to (x, y, 0)$. The homotopy $F((x, y, z), t) = (x, y, (1-t)z)$ would be a deformation retraction.

**Remark 11.11.** If $F$ is a deformation retraction, let $r(X) := F(x, 1)$. and $i : A \hookrightarrow X$ be the inclusion. Then $r \circ i = id_A$, and $i \circ r \sim id_X$.

**Example 11.12.** Any point is a deformation retract of $\mathbb{R}^n$.

**Definition 11.13.** $X$ is *homotopy-equivalent* to $Y$ ($X \sim Y$) if $\exists f : X \to Y$ and $g : Y \to X$ such that $g \circ f \sim id_X$ and $f \circ g \sim id_Y$. 

Example 11.15. For all $n, \mathbb{R}^n \sim_{\text{homotopy}}$ a point. Why? If $A$ is a deformation retract of $X$, then $A \sim_{\text{homotopy}} X$.

Theorem 11.16. If $f, g : X \to Y$ are homotopic (i.e. $f \sim g$) then $f_* = g_*$ as maps of homology $H_*(X; G) \to H_*(Y; G)$.

Corollary 11.17. If $X \sim Y$, then $H_*(X) \cong H_*(Y)$.

12. September 30, 2015

12.1. Last class. We saw what makes two maps $f, g : X \to Y$ homotopy-equivalent. We also defined homotopy-equivalent spaces to be connected by continuous maps $f : X \to Y, g : Y \to X$ such that $f \circ g = id_Y$ and $g \circ f = id_X$.

Theorem 12.1. If $f, g : X \to Y$ are homotopic (i.e. $f \sim g$) then $f_* = g_*$ as maps of homology $H_*(X; G) \to H_*(Y; G)$.

Corollary 12.2. If $X \sim Y$, then $H_*(X) \cong H_*(Y)$.

Proof. 

$$ (f \circ g)_* = id_{H_*(Y; G)} \quad (g \circ f)_* = id_{H_*(X; G)} $$

but

$$ (f \circ g)_* = f_* g_* = id_{H_*(Y; G)} \quad (g \circ f)_* = id_{H_*(Y; G)} $$

Thus $f_* = g_*^{-1}$, which means we have group isomorphism. \qed

Remark 12.3. The converse of this Theorem is not true. In particular, there exist non-homotopy equivalent spaces with isomorphic homology groups.

Example 12.4 (3-sphere). $Y =$ the Poincare homology sphere. This has

$$ H_n(Y; G) = \begin{cases} G & n = 0, 3 \\ 0 & n \notin \{0, 3\} \end{cases} $$

But $Y$ has nontrivial fundamental group $\pi_1$. In fact $\pi_1(Y)$ is the icosahedral group.

Definition 12.5. Homotopy type is an element of the category topological spaces modulo the equivalence relation of being connected by a homotopy.

Definition 12.6. $X$ is contractible if $X \sim \text{point}$.

In particular, $X$ is contractible implies $\tilde{H}_*(X) = 0$.

Remark 12.7. The converse is not true.

The homology of $X$ is determined by the homotopy type of $X$.

Let $A \subset 2^V$ be an abstract simplicial complex.

Definition 12.8. Homotopy type of $*$ is the homotopy type of its geometric realization.

Lemma 12.9. The homotopy type of $A$ does not depend on the choice of a geometric realization.

Fact 12.10. Even in the case of a finite abstract simplicial complex $A$ i.e. $A \subset 2^V$ for $|V| < \infty$, there is no algorithm deciding contractibility.

However if $\tilde{H}_*(A) \neq \emptyset$, then $A$ is not contractible.
12.2. Nerves and Cech complexes. Let $\mathcal{U} = \{U_v\}_{v \in V}$.

**Definition 12.11.** The nerve of $\mathcal{U}$ is an abstract simplicial complex $\text{nerve}(\mathcal{U}) \subset 2^V$ defined as $\text{nerve}(\mathcal{U}) = \{\sigma \subset V | \bigcap_{v \in \sigma} U_v \neq \emptyset\}$. Note that $\bigcap_{v \in \emptyset} U_v = X$.

**Notation:** $U_\sigma = \bigcap_{v \in \sigma} U_v$ is contractible.

**Remark 12.12.** This is an abstract simplicial complex i.e. $\nu \subset \sigma, \sigma \in \text{nerve}(\mathcal{U}) \implies \nu \in \text{nerve}(\mathcal{U})$.

**Definition 12.13.** The collection of sets $\mathcal{U} = \{U_v\}_{v \in V}$ is called a **locally finite cover** if:

1. $\mathcal{U}$ is a cover, i.e. $\bigcup_{v \in V} U_v = X$.
2. the cover is locally finite: $\forall x \in X$ there exists at most a finite number of $U_v$ such that $x \in U_v$.

**Theorem 12.14** (Nerve Lemma – Open Version). Assume that $\mathcal{U} = \{U_v\}_{v \in V}$ is a locally finite cover of a triangulable topological space $X$, and moreover:

1. $U_v$ are open.
2. $U_\sigma$ is contractible for all $\sigma \in \text{nerve}(\mathcal{U})$, for $\sigma \neq \emptyset$.

Then $X \sim_{\text{homotopy}} \text{nerve}(\mathcal{U})$.

**Theorem 12.15** (Nerve Lemma – Closed Version). Assume that $\mathcal{U} = \{U_v\}_{v \in V}$ is a finite cover of a triangulable topological space $X$, and moreover:

1. $U_v$ are closed.
2. $U_\sigma$ is contractible for all $\sigma \in \text{nerve}(\mathcal{U})$, for $\sigma \neq \emptyset$.

Then $X \sim_{\text{homotopy}} \text{nerve}(\mathcal{U})$.

**Example 12.16.** All open or all closed cannot be relaxed. For instance, the interval can be split into an open interval and a closed interval, which means even though the interval is contractible, it has a cover with nerve two points.

**Remark 12.17.** In the closed case, the “finite” condition cannot be dropped either.

**Example 12.18.** Consider the unit circle $X = S^1$. Let

$$U_i = \{e^{2\pi it} | \frac{1}{i+1} \leq t \leq \frac{1}{i}\}.$$ 

Claim: Homotopy type $\text{nerve}\{U_i\} \neq \text{homotopy type of } S^1$. 

Figure 6. Nerve of a set arrangement.
Example 12.19. \( S^1 \times [a, b] \sim_{hom} S^1 \).

Remark 12.20. Contractibility of every intersection: If \( X \subset \mathbb{R}^d \) is such that \( U_i \subset X \subset \mathbb{R}^d \). If \( U_i \) are convex, then any intersection is also convex! Thus it is also contractible.

\[ \text{Convex} \implies \text{contractible, since you can contract all points to a fixed point along lines.} \]