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Preface

This book is intended for a two-quarter or one or two-semester course in advanced calculus and introductory real analysis. The book is classical in the sense that it deals with calculus and Fourier series in Euclidean space. Only a few brief references are made to "modern" topics such as Lebesgue integration, distributions, and quantum mechanics. We resisted the temptation to include vector analysis (the Stokes theorem and so forth). In most curricula, this topic comes earlier in the second year at a more informal level (see, for example, J. Marsden and A. Tromba, *Vector Calculus*, W. H. Freeman and Company, 1975) and possibly later in the context of manifold theory for students who are so inclined.

In presenting the material, we have been deliberately concrete—aiming at a solid understanding of the Euclidean case and introducing abstraction only through examples. For instance, if Euclidean spaces are properly understood, it is a small jump to other spaces such as the space of continuous functions and abstract metric spaces. In the context of the space of continuous functions, we can see the power of abstract metric space methods. When the general theory is presented too soon, the student is confused about its relevance; consequently, much teaching time can be wasted.

The book assumes that the reader has had some calculus; that is, that he or she knows how to differentiate and integrate standard functions. Strictly speaking, the theory is developed logically and requires few prerequisites, although a knowledge of calculus is needed for an understanding of examples and exercises. Also, some brief contact with partial derivatives and multiple integrals is desirable but not essential. Chapter 6, on differentiation, requires the rudiments of linear algebra; specifically, the student should know what a linear transformation and its representing matrix is.

Each chapter is organized as follows. There are numerous sections containing the definitions, statements of the theorems, examples, and fairly easy problems. Once the student masters the theorems and is able to handle the easy problems, he can move on to the end of the chapter to master the technical proofs. Here, numerous further examples and exercises are given. The easier exercises following each section enable the student to master the material as he goes along. The exercises at the end of the chapter then often require an integrated knowledge of the whole chapter or previous chapters